Rsearch Internship Report for L3

Free Complete Product And Hawaiian Earring Group

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Introduction

The Hawaiian earring $\mathbb{H}$ is a subspace of the plane, defined as the union of circles with radius $1/n$ and center $(1/n, 0)$ for $n \in \mathbb{N}$. The aim of this research internship was to calculate the fundamental group of $\mathbb{H}$. It was H.B Griffiths [3] who first calculated the fundamental group of the Hawaiian earring. I began the internship by learning the fundamentals of algebraic topology, including homotopies, fundamental groups, and covering spaces, before delving into the main subject. In this report, we will not extensively cover the basics of algebraic topology but will highlight some important tools and concepts that are used.

The Hawaiian earring appears similar to the wedge sum of a countably infinite number of circles $\bigvee_{n\in\mathbb{N}} S^1$. However, we will demonstrate that this is not the case and calculate the fundamental group of the wedge sum of infinite circles using Van Kampen's theorem, finding it to be the free product of countably infinite copies of $\mathbb{Z}$, $*_{n\in\mathbb{N}}\mathbb{Z}$. We will also show that the fundamental groups of these two spaces are different.

In the second section, we will introduce and explain the concept of free complete products of groups, which is a generalization of free products of groups where the word can be of infinite length. This notion is presented by Eda [1]. In the third section, we will calculate the fundamental group of $\mathbb{H}$, finding it to be the free complete product of countably infinite copies of $\mathbb{Z}$. Additionally, we will show that $\pi_1(\mathbb{H})$ embeds into an inverse limit of free groups.

For prerequisites, readers should be familiar with general topology, groups, homotopies, the fundamental group, free products of groups, and free groups. We will use also transfinite induction in the existence part of the proof of theorem 2.4, and Zorn's lemma in the proof of proposition 2.6.

Throughout this report, $\mathbb{N}$ denotes the set of positive integers, $\{1, 2, \ldots\}$.

A set is said to be countable if it is either finite or countably infinite.

When we state that a property $P(i)$ holds for almost all $i \in I$, we mean that $P(i)$ is true for all but finitely many i in the set I.

1 Foundational Concepts

1.1 Some Basics of Algebraic topology

Definition 1.1. *Given a topological space X, a **covering space** of X consists of a topological space $\tilde{X}$ and a map $p : \tilde{X} \to X$ satisfying the following condition: For each point $x \in X$ there is an open neighborhood U of x in X such that $p^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by p. We remark that p is necessary continuous.*

Definition 1.2. *Let X, Y, Z be three topological spaces and $f : X \to Y$ and $g : Z \to Y$ be two continuous maps. A **lift or lifting** of f to Z is a continuous map $h : X \to Z$ such that $f = g \circ h$.*

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ X \arrow[ur, "h"] \arrow[r, "f"] & Y \end{tikzcd}$$

Homotopy lifting property:

Given E, B, Y topological spaces, a continuous map $\pi : E \to B$, we say that (Y, π) has the homotopy lifting property, or π has the homotopy lifting property with respect to Y, if for any homotopy $f_\bullet : Y \times I \to B$ and any map $\tilde{f}_0 : Y \to E$ lifting $f_0 = f_\bullet|_{Y\times\{0\}}$, there is a homotopy $\tilde{f}_\bullet : Y \times I \to E$ lifting f , such that $\tilde{f}_0 = \tilde{f}\Big|_{Y\times\{0\}}$. And if $\tilde{f}_0$ is unique we say that (Y, π) has the unique homotopy lifting property.

$$\begin{tikzcd} Y \arrow[r, "\tilde{f}_0"] \arrow[d, hook, "\iota_0"] & E \arrow[d, "\pi"] \\ Y \times I \arrow[r, "f_\bullet"] \arrow[ur, dashed, "\tilde{f}_\bullet"] & B \end{tikzcd}$$

Theorem 1.3. *Let Y, X be topological spaces and $\tilde{X}, p : \tilde{X} \to X$ a covering space of X, then (Y, p) has the unique homotopy lifting property.*

Corollary 1.4. *(a) For each path $f : I \to X$ starting at a point $x_0 \in X$ and for each $\tilde{x_0} \in p^{-1}(x_0)$ there is a unique lift $\tilde{f} : I \to \tilde{X}$ starting at $\tilde{x_0}$.*

(b) For each homotopy $f_t : I \to X$ of paths starting at x_0 and each $\tilde{x_0} \in p^{-1}(x_0)$ there is a unique lifted homotopy $\tilde{f}_t : I \to \tilde{X}$ of paths starting at $\tilde{x_0}$.

Theorem 1.5. *The fundamental group of the circle is cyclic infinite. $\pi_1(S^1) \cong \mathbb{Z}$.*

Calculation of $\pi_1(S^1)$ has many applications. For example: the fundamental theorem of algebra, the Brower fixed point theorem in dimension 2 and the Borsuk Ulam theorem in dimension 2.

Proposition 1.6. *If X and Y are path connected topological spaces then $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$.*

Example 1.7. *The fundamental group of the torus: $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$.*

Proposition 1.8. *Let $\varphi : X \to Y$ be a continuous map with $\varphi(x_0) = y_0$, we write then $\varphi : (X, x_0) \to (Y, y_0)$. Then φ induces a homomorphism: $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by $\varphi_*([f]) = [\varphi \circ f]$ for $[f] \in \pi_1(X, x_0)$, Well, it's well defined since for any homotopy F based at x_0, we have $\varphi \circ F$ is a homotopy based at y_0. And it is a homomorphism since $\varphi(f.g) = (\varphi \circ f).(\varphi \circ g)$.*

And we have those evident properties:

1. $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$

2. $1_* = 1$

3. If φ is a homeomorphism with inverse ψ then φ_* is an isomorphism with inverse ψ_*.

Proposition 1.9. *For $n \geq 2$, $\pi_1(S^n) = 0$.*

As a corollary of this proposition we have:

Corollary 1.10. *$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for a given positive integer $n \neq 2$.*

Definition 1.11. *Let X be a topological space and $A \subseteq X$. Then $r : X \to A$ is a* ***retraction*** *if it's continuous and $r|_A = id_A$.*

A ***deformation retraction*** *is a homotopy between a retraction and the identity map on X that satisfies for all $t \in I, F(\bullet, t)_{|A} = id_A$, that is a continuous map $F : X \times [0, 1] \to X$ such that $\forall x \in X, \forall a \in A$,*

$$F(x, 0) = x, \quad F(x, 1) \in A, \quad \text{and} \quad F(a, t) = a.$$

And then the subspace A is called a deformation retract of X.

Proposition 1.12. *If a space X retracts onto a subspace A, then the homomorphism $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $i : A \to X$ is injective. If A is a deformation retract of X, then i_* is an isomorphism.*

Definition 1.13. *A space X is called* ***simply connected*** *if it is path connected and has trivial fundamental group*

Proposition 1.14. *A space X is simply connected if and only if there is a unique homotopy class of paths connecting any two points in X.*

Definition 1.15. *A space X is said to be* ***locally simply connected*** *at a point $x \in X$ if for every neighborhood V of x there is an open neighborhood $U \subset V$ of x that is simply connected. And X is said to be locally simply connected if X is locally simply connected at each point $x \in X$.*

For proofs of the previous propositions and more details, you can check [2].

1.2 Van Kampen's theorem

Let (G_α) be a family of groups, then any collection of group homomorphisms $\varphi_\alpha : G_\alpha \to H$ extends uniquely to a homomorphism $\varphi : *_\alpha G_\alpha \to H$. For a word $g_1 \dots g_n$ with $g_i \in G_{\alpha_i}, \varphi(g_1 \dots g_n) = \varphi_{\alpha_1}(g_1) \dots \varphi_{\alpha_n}(g_n)$. It is easy to see that φ is well defined and a homomorphism.

Suppose a topological space X is decomposed as the union of a collection of path-connected open subsets A_α, each of which contains the basepoint $x_0 \in X$. By the remark above the homomorphisms $j_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$ induced by the inclusions $A_\alpha \hookrightarrow X$ extend to a homomorphism $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$. If $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ is the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ then $j_\alpha \circ i_{\alpha\beta} = j_\beta \circ i_{\beta\alpha}$, indeed, using property 1 of proposition 1.8 we see that both $j_\alpha \circ i_{\alpha\beta}$ and $j_\beta \circ i_{\beta\alpha}$ are induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow X$. Then $\Phi(i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}) = j_\alpha \circ i_{\alpha\beta}(\omega) j_\beta i_{\beta\alpha}(\omega)^{-1} = j_\alpha \circ i_{\alpha\beta}(\omega\omega^{-1}) = 0$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, so for all $\omega \in \pi_1(A_\alpha \cap A_\beta)$, $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1} \in Ker(\Phi)$.

Theorem 1.16 (Van Kampen's theorem). *If X is the union of path-connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective. If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and hence Φ induces an isomorphism $\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)/N$.*

A proof of this theorem can be found in [2].

Remark 1.17. *In the special case where $X = A_\alpha \cup A_\beta$ and A_α, A_β are open path connected subsets of X containing x_0 and $A_\alpha \cap A_\beta$ is path connected, then the condition: "each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected" is automatically satisfied, so $\pi_1(X) \cong (\pi_1(A_\alpha) * \pi_1(A_\beta))/N$.*

Definition 1.18 (Wedge Sum). *Let $(X_i)_{i \in J}$ be a family of topological spaces, and $(x_i)_{i \in J} \in \prod_{i \in J} X_i$, we then call the couple (X_i, x_i) pointed space with base point x_i, the wedge sum of the family $(X_i, x_i)_{i \in J}$ is the set $\bigvee_{i \in J} X_i ;= \coprod_{i \in J} X_i \,/\!\sim$ equipped with the quotient topology of the disjoint union topology of $(X_i)_{i \in J}$ over the relation $\sim$ that is defined by: $\forall i, j \in J, x_i \sim x_j$. We will identify $[x]$ by x for $x \notin (x_i)_{i \in I}$ and $[x_i]$ by x_i for $i \in J$ so $x_i = x_j$ for $i, j \in J$.*

For example: the space $S^1 \bigvee S^1$ is homeomorphic to the figure eight in the plane.

An application of Van Kampen's theorem is the calculation of the fundamental group of the wedge sum, which is stated in the following theorem.

Theorem 1.19. *Let $\bigvee_\alpha X_\alpha$ be the wedge sum of the pointed spaces (X_α) with the base points (x_α). If for each α, X_α is path connected, and x_α is a deformation retract of an open neighborhood U_α in X_α then $\pi_1(\bigvee_\alpha X_\alpha) \cong *_\alpha \pi_1(X_\alpha)$.*

Proof. For each α, there is $F_\alpha : U_\alpha \times I \to U_\alpha$ continuous such that:

$$F(x, 0) = x, F(x_\alpha, t) = x_\alpha, F(x, 1) = x_\alpha, \forall x \in U_\alpha, \forall t \in I$$

Fix α. Pose $A_\alpha = X_\alpha \bigvee_{\beta \neq \alpha} U_\beta$. A_α is obviously open in $\bigvee_\alpha X_\alpha$. Consider the map $H_\alpha : A_\alpha \times I \to A_\alpha$ defined by:

$$H_\alpha(b, t) = F_\beta(b, t), \forall \beta \neq \alpha, \forall b \in U_\beta, \forall t \in I \text{ and } H_\alpha(x, t) = x, \forall x \in X_\alpha, \forall t \in I$$

We have for all $\beta \neq \alpha$, for all $t \in I$, $H_\alpha(x_\beta, t) = F_\beta(x_\beta, t) = x_\beta = x_\alpha = H_\alpha(x_\alpha, t)$ this verifies that H_α is well defined.

Let's show that H_α is continuous. Given $(x,t) \in A_\alpha \times I$, and V an open neighborhood of $H(x,t)$.

Suppose firstly that $x_\alpha \notin V$. If $x \in X_\alpha$, then $V \cap X_\alpha$ is an open neighborhood of x in X_α, and since $x_\alpha \notin V \cap X_\alpha$, then $V \cap X_\alpha$ is an open neighborhood of x in A_α. And, $H_\alpha(V \cap X_\alpha \times I) \subset V$. If $x \in U_\beta$ for some $\beta \neq \alpha$, then $F_\beta^{-1}(V \cap U_\beta)$ is an open neighborhood of (x,t) in $U_\beta \times I$, so there is W open of U_β and T open of I such that $(x,t) \in W \times T \subset F_\beta^{-1}(V \cap U_\beta)$. And since $x_\beta \notin W$ then W is open in A_α, and we have that $H_\alpha(W \times T) \subset V$.

Suppose now that $x_\alpha \in V$. Suppose in the first subcase that $x \in X_\alpha$. Given $\beta \neq \alpha$, we have for all $t \in I$, $(x_\beta, t) \in F_\beta^{-1}(V \cap U_\beta)$ which is open. So, for all $t \in I$, there is I_t open neighborhood of t in I, and there is W_t open neighborhood of x_β in U_β, such that $W_t \times I_t \subset F_\beta^{-1}(V \cap U_\beta)$. $(I_t)_{t \in I}$ is an open cover of I which is compact, so there is $t_1, \ldots, t_k \in I$ such that $I = \bigcup_{n=1}^k I_{t_n}$. Pose $W_\beta = \bigcap_{n=1}^k W_{t_n}$, which is open neighborhood of x_β in U_β and verifies $W_\beta \times I \subset F_\beta^{-1}(V \cap U_\beta)$. Pose $W_\alpha = V \cap X_\alpha$. Then, $W := W_\alpha \bigvee_{\beta \neq \alpha} W_\beta$ is open neighborhood of x in A_α and $H(W \times I) \subset V$.

Suppose in the second subcase that $x \in U_\gamma$ for some $\gamma \neq \alpha$. Similarly as for the previous paragraph for each $\beta \notin \{\alpha, \gamma\}$ there is W_β open neighborhood of x_β in U_β such that $W_\beta \times I \subset F_\beta^{-1}(V \cap U_\beta)$. Pose $W_\alpha = V \cap X_\alpha$. There is W_γ open neighborhood of x in X_γ and J open neighborhood of t in I such that $W_\gamma \times J \subset F_\gamma^{-1}(V \cap U_\gamma)$. If $x_\gamma \notin W_\gamma$ then W_γ is open neighborhood of A_α and we have $H(W_\gamma \times J) \subset V$. Suppose that $x_\gamma \in W_\gamma$. Then, $W := W_\alpha \bigvee W_\gamma \bigvee_{\beta \notin \{\alpha,\gamma\}} W_\beta$ is an open neighborhood of x in A_α and $H(W \times J) \subset V$.

We conclude then that H_α is continuous. Hence, for each α, X_α is a deformation retract of its open neighborhood A_α. So, by proposition 1.12, we have $\pi_1(A_\alpha, x_\alpha) \cong \pi_1(X_\alpha, x_\alpha)$. The intersection of two or more distinct A_α's is $\bigvee_\alpha U_\alpha$ which deformation retracts to a point, it is easy to check this working similarly as above. Then, $\bigvee_\alpha U_\alpha$ is path connected and has trivial fundamental group. We have for each α, U_α deformation retracts to a point, so it is path connected. Then for each α, A_α is path connected.

Van Kampen's theorem then implies that $\Phi : *_\alpha \pi_1(X_\alpha) \to \pi_1(\bigvee_\alpha X_\alpha)$ is an isomorphism.

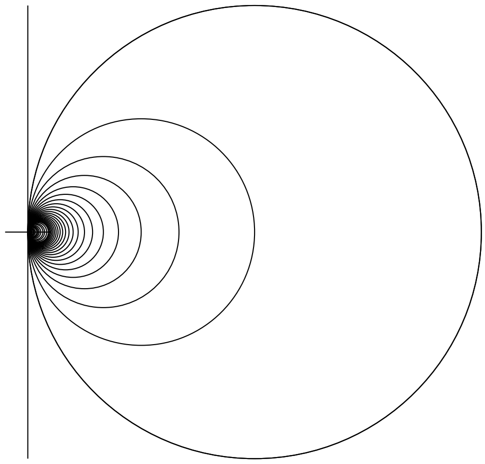
□

Example 1.20. $\pi_1(\bigvee_\alpha S^1) \cong *_\alpha \mathbb{Z}$.

1.3 The Hawaiian earring

the **Hawaiian earring** $\mathbb{H}$ is the topological space defined by union of the circles C_n in the euclidean plane $\mathbb{R}^2$ of radius $\frac{1}{n}$ and center $(\frac{1}{n}, 0)$ for $n \in \mathbb{N}$ endowed with the subspace topology.

$$\mathbb{H} = \bigcup_{n=1}^{\infty} \left\{ (x,y) \in \mathbb{R}^2 \mid \left(x - \frac{1}{n}\right)^2 + y^2 = \left(\frac{1}{n}\right)^2 \right\}$$



The Hawaiian earring $\mathbb{H}$ looks very similar to the wedge sum of countably infinitely many circles $\bigvee_{i=1}^{\infty} S^1$, but this is not the case, consider $(x_i)_{i=1}^{\infty}$ the base points of $(S_i^1)_{i=1}^{\infty}$ copies of S^1 and $\bigvee_{i=1}^{\infty} S^1$ the wedge sum of $(S_i^1, x_i)_{i=1}^{\infty}$. For each $i \in \mathbb{N}$ let L_i be an open arc neighborhood of x_i in S_i^1, then $\bigvee_{i=1}^{\infty} L_i$ is open neighborhood of x_0 in $\bigvee_{i=1}^{\infty} S^1$, but any open neighborhood of the origin in $\mathbb{H}$ contains all but finitely circles C_i, this shows that we cannot see $\mathbb{H}$ as $\bigvee_{i=1}^{\infty} S^1$ in the natural way. Let's prove that X and $\bigvee_{n=1}^{\infty} S^1$ are topologically different (i.e not homeomorphic). $\mathbb{H}$ is closed in $\mathbb{R}^2$ because its complementary is union of intersection of an interior of a disk and an exterior of disk which is open, $\mathbb{H}$ is bounded so it is compact, while $\bigvee_{i=1}^{\infty} S^1$ is not compact, indeed: let $\pi : \coprod_{i=1}^{\infty} S_i^1 \to \bigvee_{i=1}^{\infty} S_i^1$ be the canonical projection, and $U_n = \pi(S_n^1 \coprod_{i \neq n} L_i)$ where L_i is the open semi circle of S_i^1 neighborhood of x_i, then $(U_n)_{n \in \mathbb{N}}$ is an open cover of $\bigvee_{i=1}^{\infty} S_i^1$ which doesn't have a finite open cover.

Proposition 1.21. *The Hawaiian earring $\mathbb{H}$ is the one point Compactification (i.e Alexandroff) of a countable disjoint union of open intervals endowed with the disjoint union topology.*

Proof. Countable disjoint union of open intervals endowed with the disjoint union topology is homeomorphic to $\mathbb{H} \setminus \{O\}$ where O is the origin of the plane, $id : \mathbb{H} \setminus \{O\} \to \mathbb{H}$ is trivially a homeomorphism from $\mathbb{H} \setminus \{O\}$ to $\mathbb{H} \setminus \{O\}$, then by Alexandroff compactification theorem and since $\mathbb{H}$ is compact then $\mathbb{H}$ is the one point compactification of $\mathbb{H} \setminus \{O\}$ so the one point compactification of countable disjoint union of open intervals. $\square$

About the fundamental group of Hawaiian earring

The Hawaiian earring $\mathbb{H}$ is path connected, so its fundamental group doesn't depend on the choice of the basepoint, we take the origin as the basepoint. we show that $\mathbb{H}$ has a much larger fundamental group than the wedge sum. Consider the maps $r_n : X \to C_n$ collapsing all C_i's except C_n to the origin, and conserves C_n, they are trivially continuous so they are retractions. Each r_n induces a surjection $\rho_n : \pi_1(X) \to \pi_1(C_n) \approx \mathbb{Z}$ indeed we have $r_n \circ i = id$ so $r_{n*} \circ i_* = id$ we take $\rho_n = r_{n*}$. The product of the ρ_n's is a homomorphism $\rho : \pi_1(X) \to \prod_{i=1}^{\infty} \mathbb{Z}$ to the direct product of infinitely many copies of $\mathbb{Z}$, and ρ is surjective since for every sequence of integers k_n we can construct a loop $f : I \to X$ that wraps k_n times around C_n in the time interval $[1 - 1/n, 1 - 1/(n+1)]$.

This infinite composition of loops is certainly continuous at each time less than 1, and it is continuous at time 1 since every neighborhood of the basepoint in $\mathbb{H}$ contains all but finitely many of the circles C_n. Since $\pi_1(\mathbb{H})$ maps onto (i.e surjectively) the uncountable group $\prod_{i=1}^{\infty}\mathbb{Z}$, it is uncountable. On the other hand, the fundamental group of a wedge sum of countably many circles is countably generated, hence countable.

Proposition 1.22. *1. $\pi_1(\mathbb{H})$ is uncountable.*

2. For each $n \in \mathbb{N}$, $*_{i=1}^{n}\mathbb{Z}$ is a proper subgroup of $\pi_1(\mathbb{H})$. In particular $\pi_1(\mathbb{H})$ is non abelian, so it's more complicated than $\prod_{i=1}^{\infty}\mathbb{Z}$.

3. $\mathbb{H}$ is locally path connected and not locally simply connected.

Proof. 1. It has been proven above.
2. Consider the retraction $r : \mathbb{H} \to C_1 \cup \cdots \cup C_n$ that collapses all the circles smaller than C_n to the basepoint, then by 1.12 $i_* : \pi_1(C_1 \cup \cdots \cup C_n) \to \pi_1(\mathbb{H})$ is injective, so $*_{i=1}^{n}\mathbb{Z} \cong \pi_1(C_1 \cup \cdots \cup C_n)$ is isomorphic to a subgroup of $\pi_1(\mathbb{H})$.
3. $\mathbb{H}$ is trivially locally path connected, every neighborhood of the origine O, contains all but a finite number of circles C_n, collapse all the neighborhood to O except one circle, then this is a retraction, hence $\mathbb{Z}$ is subgroup of the fundamental group of this neighborhood, so it is not trivial. □

2 Free Complete Products

2.1 Definition and properties

Definition 2.1. *Let $(G_i)_{i\in I}$ be a family of groups. We assume $G_i \cap G_j = \{e\}$ for distinct $i, j \in I$. Elements of $\bigcup_{i\in I} G_i$ are called letters. W is a word, if W is a function from a totally ordered set $\overline{W}$ to $\bigcup_{i\in I} G_i$, such that $W^{-1}(G_i)$ is finite for each i. In case the cardinality of $\overline{W}$ is countable, we say that W is a σ-word. The class of all words is denoted by $\mathcal{W}(G_i : i \in I)$ (abbreviated by $\mathcal{W}$) and the class of all σ-words is denoted by $\mathcal{W}_\sigma(G_i : i \in I)$ (abbreviated by $\mathcal{W}_\sigma$).*

For given $U, V \in \mathcal{W}$, we say that U and V are isomorphic and denote it by $U \simeq V$, If there exists a bijective map that preserves the order $i : \overline{U} \to \overline{V}$ and $U(\alpha) = V(i(\alpha))$ for all $\alpha \in \overline{U}$. Then $\simeq$ is an equivalence relation on the class $\mathcal{W}$. Since the cardinality of $\overline{W}$ is less than or equal to $\max\{|I|, \aleph_0\}$ for a word W, then the class $\mathcal{W}/\simeq$ is a set, we then identify the equivalence class of a word W with W and $\mathcal{W}/\simeq$ with $\mathcal{W}$. For words of finite length, this is the same as the usual definition of words in the free product of groups. For a word $W \in \mathcal{W}(G_i : i \in I)$ and a subset $X \subset I$, W_X is the word obtained by eliminating letters not in $\bigcup_{i\in X} G_i$; that is, $W_X \in \mathcal{W}(G_i : i \in X)$, $\overline{W_X} = \{\alpha \in \overline{W} : W(\alpha) \in \bigcup_{i\in X} G_i\}$ equipped with the restricted order of $\overline{W}$ on it , and $W_X(\alpha) = W(\alpha)$ for $\alpha \in \overline{W}$. For words U and V, we say that $U \sim V$ holds if $U_F = V_F$ for every $F \subset I$ finite, where we regard U_F, V_F as elements of the free product $*_{i\in F}G_i$. Then, $\sim$ is well defined relation and it is an equivalence relation on $\mathcal{W}$ clearly. Denote the equivalence class containing U by $[U]$. For $U, V \in \mathcal{W}$, let UV be the composition of U and V, that is, $\overline{UV} = \{(0, \alpha), (1, \beta) : \alpha \in \overline{U}, \beta \in \overline{V}\}$, where $(0, \alpha) < (1, \beta)$ for $\alpha \in \overline{U}$ and $\beta \in \overline{V}$ and $(i, \alpha) < (i, \beta)$ for $\alpha < \beta$ and $i = 0, 1$; $UV((0, \alpha)) = U(\alpha)$ and $UV((1, \beta)) = V(\beta)$. Let U^{-1} be the word such that $\overline{U^{-1}} = \{(0, \alpha) : \alpha \in \overline{U}\}$, where

$(0,\alpha) < (0,\beta)$ if $\alpha > \beta$ and $U^{-1}((0,\alpha)) = U(\alpha)^{-1}$. Then, it is easy to see that the operation $[U][V] := [UV]$ on $\mathcal{W}/\sim = \{[W] : W \in \mathcal{W}\}$ is well defined and $\mathcal{W}/\sim$ becomes a group with this operation, he neutral element is the class of the empty word and $[U]^{-1} = [U^{-1}]$.

Definition 2.2. *The free complete product* $\times_{i\in I} G_i$ *is the group* $\mathcal{W}(G_i : i \in I)/\sim$. *The free* σ*-product* $\times^{\sigma}_{i\in I} G_i$ *is the group* $\mathcal{W}_\sigma(G_i : i \in I)/\sim$, *which is a subgroup of* $\times_{i\in I} G_i$. *In case every* G_i *is isomorphic to* G, *we abbreviate* $\times_{i\in I} G_i$ *by* $\times_I G$ *and similarly for free* σ*-products.*

Obviously the free product $*_{i\in I} G_i$ is a subgroup of $\times^{\sigma}_{i\in I} G_i$ so a subgroup of $\times_{i\in I} G_i$, and if I is finite then they are all isomorphic.

Definition 2.3. *A word* W *is said to be reduced, if* $W \simeq UXV$ *implies* $[X] \neq e$ *for any non-empty word* X, *where* e *is the identity, and for any neighboring elements* α *and* β *of* $\overline{W}$ *it never occurs that* $W(\alpha)$ *and* $W(\beta)$ *belong to the same* G_i. α *and* β *are said to be neighboring elements if for all* γ *verifies* $\alpha \le \gamma \le \beta$ *or* $\alpha \ge \gamma \ge \beta$ *then* $\gamma = \alpha$ *or* $\gamma = \beta$.

Theorem 2.4. *For any word* W, *there exists a reduced word* V *such that* $[W] = [V]$ *and* V *is unique up to isomorphism.*

We will present Eda's proof [1], which employs the transfinite recursion for the existence part.

Proof. The Existence: We define words W_μ for ordinals μ by induction. Let W_0 be W. If there exists a non-empty word X such that W_μ is isomorphic to UXV and $[X] = e$, let $\overline{W_{\mu+1}} = \{\alpha \in \overline{W_\mu} : i(\alpha) \in \overline{U} \text{ or } i(\alpha) \in \overline{V}\} \subset \overline{W}$ and $W_{\mu+1}(\alpha) = W(\alpha)$ for $\alpha \in \overline{W_{\mu+1}}$, where the ordering is the restriction of that of $\overline{W}$ and $i : \overline{W_\mu} \to \overline{UXV}$ is the order isomorphism. Otherwise, the procedure is completed. For a limit ordinal μ, let $\overline{W_\mu} = \bigcap_{\nu<\mu} \overline{W_\nu}$ and $W_\mu(\alpha) = W(\alpha)$ for $\alpha \in \overline{W_\mu}$. This procedure must stop at some ordinal whose cardinality is at most $\max\{|I|, \aleph_0\}$ because the cardinality of $\overline{W}$ is equal to or less than $\max\{|I|, \aleph_0\}$. Let W_∞ be the obtained word. By induction, we can see that $[W_\mu] = [W]$ and hence $[W_\infty] = [W]$. There may be a neighboring $\alpha, \beta \in \overline{W_\infty}$ such that $W_\infty(\alpha)$ and $W_\infty(\beta)$ belong to the same G_i. Since such occasions happen only finitely many times for each i, performing the calculation in each G_i we obtain the desired reduced word of W.

The uniqueness: suppose that $[U] = [V]$ for reduced words U and V. We define $\varphi : \overline{U} \to \overline{V}$ in the following manner. For $\alpha \in \overline{U}$ there exists a unique $i \in I$ such that $U(\alpha) \in G_i$. Then, there is $g_1, \ldots, g_m \in G_i$, and $X_1 \cdots X_{m+1} \in \mathcal{W}(G_j : i \neq j \in I)$ such that $U \simeq X_1 g_1 X_2 \cdots X_m g_m X_{m+1}$, since U is reduced then $\forall k \in \{2, 3, \ldots m\}, [X_k] \neq e$, so there is $E_2, E_3, \ldots E_m \subset I$ finite such that $[(X_k)_{E_k}] \neq e$, take $E = E_2 \cup E_3 \cup \ldots \cup E_m \cup \{i\}$ then $\forall k \in \{2, 3, \ldots, m\}, [(X_k)_E] \neq e$, and $U_E \simeq (X_1)_E g_1 (X_2)_E \cdots (X_m)_E g_m (X_{m+1})_E$ and $U(\alpha) = g_k$ for some $k \in \{1, \ldots m\}$. Similarly there exist $F \subset I$ finite, letters $g'_1, \ldots, g'_n \in G_i$, and $Y_1, \cdots, Y_{n+1} \in \mathcal{W}(G_j : i \neq j \in I)$ such that $V \simeq Y_1 g'_1 Y_2 \cdots Y_n g'_n Y_{n+1}$, $V_F \simeq (Y_1)_F g'_1 (Y_2)_F \cdots (Y_n)_F g'_n (Y_{n+1})_F$, and $\forall k \in \{2, 3, \ldots, n\}, [(Y_k)_E] \neq e$. Because we can take F and E as bigger as we want and as they are finite, we can suppose $E = F$. Since $[U_F] = [V_F]$ then, $m = n$ and $g(l) = g'(l)$ for $1 \leq l \leq m$. Let $\varphi(\alpha) \in \overline{V}$ be the member corresponding to g'_k in V. Clearly φ is a bijective map and $U(\alpha) = V(\varphi(\alpha))$. Taking large enough $F \subset I$ as the above, we can see that φ preserves the order. Therefore, U and V are isomorphic. $\square$

From now on we regard a word as an element of $\times_{i \in I} G_i$ so $U = V$ means $[U] = [V]$ for words U and V.

Corollary 2.5. *Let U and V be reduced words. If $UV = e$, then V is isomorphic to U^{-1}.*

Proof. $e = [UV] = [U][V]$, so $[U^{-1}] = [V]$, and since U is reduced then U^{-1} is also reduced. Indeed, if $U^{-1} \simeq W_1 X W_2$ with X non empty word, then $U \simeq W_2^{-1} X^{-1} W_1^{-1}$ and X^{-1} is non empty, since U is reduced then $[X^{-1}] \neq e$ so $[X] \neq e$. We conclude then by uniqueness part of the theorem that V is isomorphic to U^{-1}. $\square$

Proposition 2.6. *Let U be a reduced word.*

1. *There exists no nonempty reduced word X such that $U = UX$ or $U = XU$.*
2. *If U is nonempty and $U = U^{-1}$, then there exist a reduced word X and a letter g such that U is isomorphic to $X^{-1}gX$ and $g^2 = e$.*

Proof. 1. $[U] = [UX] = [U][X]$ implies $[X] = e$, and since X is reduced then by uniqueness part of the theorem X is isomorphic to the empty word, so it is empty.

2. Since U^{-1} is also reduced, $U = U^{-1}$ implies $U \simeq U^{-1}$ and, hence, let $i : \overline{U} \to \overline{U^{-1}}$ be the order isomorphism. Let $\mathcal{A} = \{A \subset \overline{U} :$ if $\alpha > \beta \in A$ then $\alpha \in A$, and $\forall \alpha \in A, i^{-1}(0, \alpha) \notin A\}$. Let's show using Zorn lemma that $\mathcal{A}$ admits a maximal element. $\emptyset \in \mathcal{A}$, so $\mathcal{A}$ is nonempty. Let $(A_i)_{i \in I}$ be a totally ordered subset of $\mathcal{A}$, take $A = \cup_{i \in I} A_i$, let $\alpha > \beta \in A$, then $\beta \in A_i$ for some $i \in I$, so $\alpha \in A_i \subset A$. Given $\alpha \in A$, suppose by absurd that $i^{-1}(0, \alpha) \in A$, so $i^{-1}(0, \alpha) \in A_i$ for some $i \in I$, and we have that $\alpha \in A_j$ for some $j \in I$, $(A_i)_{i \in I}$ s totally ordered, so $A_i \subset A_j$ or $A_j \subset A_i$ and then we get that both of α and $i^{-1}(0, \alpha)$ belong to A_j or A_i, which is contradiction. Hence $A \in \mathcal{A}$ and A is an upper bound of $(A_i)_{i \in I}$. We conclude by Zorn lemma that $\mathcal{A}$ admits a maximal element, let it be $\overline{X}$, let's equip $\overline{X}$ with the restricted order of U, and define the word X, for all $\alpha \in \overline{X}, X(\alpha) = U(\alpha)$.

Suppose $\overline{X} \cup i^{-1}\{(0, \alpha) : \alpha \in \overline{X}\} = \overline{U}$. Let $F \subset I$ finite, then $\overline{X_F} = \emptyset$ or $= \{\alpha_1 < \alpha_2 < \cdots < \alpha_k\}$ for some $k \in \mathbb{N}$ and $\alpha_1, \cdots, \alpha_k \in \overline{X}$. In the first case $\overline{U_F} = \emptyset$. In the second case $\overline{U_F} = \{i^{-1}(0, \alpha_k) < \cdots < i^{-1}(0, \alpha_1) < \alpha_1 < \cdots < \alpha_k\}$, indeed we have $\alpha_i > i^{-1}(0, \alpha_j)$ since otherwise we obtain $i^{-1}(0, \alpha_j) \in \overline{X}$ which is absurd, and $\alpha_1 < \cdots < \alpha_k$ implies $(0, \alpha_k) < \cdots < (0, \alpha_1)$ in $\overline{U^{-1}}$, so $i^{-1}(0, \alpha_k) < \cdots < i^{-1}(0, \alpha_1)$. And because of $U(i^{-1}(0, \alpha_i)) = U^{-1}(0, \alpha_i) = U(\alpha_i)^{-1}$ then $U_F = 0$, and then that $U = 0$, but U is nonempty and reduced so $U \neq 0$. We obtain then a contradiction. Hence, $\overline{X} \cup i^{-1}\{(0, \alpha) : \alpha \in \overline{X}\} \neq \overline{U}$.

Let $\overline{V} = \overline{U} \setminus (\overline{X} \cup i^{-1}\{(0,\alpha) : \alpha \in \overline{X}\})$ equipped with the restricted order of $\overline{U}$. We have obviously for all $\beta \in \overline{V}$, for all $\alpha \in \overline{X}, \beta < \alpha$. Suppose by absurd that $\beta \leq i^{-1}(0,\alpha)$ for some $\beta \in \overline{V}$ and $\alpha \in \overline{X}$, then $(0,\gamma) = i(\beta) \leq (0,\alpha)$ for some $\gamma \in \overline{U}$, then $\gamma \geq \alpha$, so $\gamma \in \overline{X}$, then $\beta = i^{-1}(0,\gamma) \in i^{-1}\{(0,\alpha) : \alpha \in \overline{X}\}$, which is absurd. Hence, $U \simeq X^{-1}VX$ with $V = U|_{\overline{V}}$.

Suppose V is not a unique letter.

Then, there is $\alpha \in \overline{V}$ such that $i^{-1}(0,\alpha) < \alpha$. Indeed, suppose not, then fix $\alpha \in \overline{V}$. Suppose in the first case that $\alpha = i^{-1}(0,\alpha)$, if there is $\beta \in \overline{V}$ such that $\beta > \alpha$ then $\beta > i^{-1}(0,\beta)$ which is absurd. Then there is $\beta \in \overline{V}$ such that $\beta < \alpha$, let $\gamma \in \overline{U}$ such that $i^{-1}(0,\gamma) = \beta$, then $\gamma > \alpha$, and since $\beta \notin i^{-1}\{(0,\alpha) : \alpha \in \overline{X}\}$, then $\gamma \in \overline{V}$, and then we get an absurd. Suppose now that $\alpha < i^{-1}(0,\alpha)$. Then for all $\beta < \alpha$ we have $\alpha < i^{-1}(0,\alpha) < i^{-1}(0,\beta)$. And for $\beta \in \overline{V}$ that verifies $\beta > \alpha$ we have $\alpha < \beta \leq i^{-1}(0,\beta)$. And for all $\beta \in \overline{X}$ we have $i^{-1}(0,\beta) < \alpha$. Hence, there is no $\beta \in \overline{U}$ such that $\alpha = i^{-1}(0,\beta)$. which is absurd.

Then, take $A = \{\beta \in \overline{U} : \beta \geq \alpha\}$. we have $A \in \mathcal{A}$ and contains strictly $\overline{X}$, which is contradiction with maximality of $\overline{X}$. Hence, $V = g$ for some letter g, and since V is reduced then $g \neq e$. Hence, $U \simeq X^{-1}gX$ and by $U \simeq U^{-1}$, we get $g^2 = e$.

□

2.2 Inverse system and Inverse limit

Definition 2.7. ***Inverse system*** *Let $(I,\leq)$ be an ordered set, let $(A_i)_{i\in I}$ be a family of groups and suppose we have a family of morphisms: $f_{ij} : A_j \to A_i$ for all $i \leq j$ in I with the following properties:*

1. *f_{ii} is the identity on A_i.*
2. *$f_{ik} = f_{ij} \circ f_{jk}$ for all $i \leq j \leq k$.*

Then the pair $((A_i)_{i\in I}, (f_{ij})_{i\leq j, i,j\in I})$ is called an inverse system of groups and morphisms over I. The morphisms (f_{ij}) are called the transition morphisms of the system.

Definition 2.8. ***Inverse limit*** *of the inverse system $((A_i)_{i\in I}, (f_{ij})_{i\leq j,i,j\in I})$ is defined as the following particular subgroup of the direct product of the sets A_i:*

$$\varprojlim (A_i, f_{ij} : i \leq j, i, j \in I) := \left\{ a \in \prod_{i\in I} A_i \;\middle|\; a_i = f_{ij}(a_j) \text{ for all } i \leq j \text{ in } I \right\}.$$

it is easy to see that A is a subgroup of the direct product $\prod_{i\in I} A_i$.

Let's return to our context, and to the notation before Definition 2.7, and let's denote by $F \Subset I$ for a finite subset F of I.

Let $p_{XY} : *_{i\in Y} G_i \to *_{i\in X} G_i$ be the canonical homomorphisme for $X \subset Y \subset I$.

Proposition 2.9. *The free complete product $\times_{i\in I} G_i$ is isomorphic to*

$$\bigcap_{F \Subset I} *_{i\in F} G_i * \varprojlim (*_{i\in X} G_i, p_{XY} : X \subset Y \Subset I \setminus F)$$

which is a subgroup of

$$\varprojlim(*_{i\in X}G_i, p_{XY} : X \subset Y \Subset I).$$

Proof. First, let's justify the writing $\bigcap_{F\Subset I} *_{i\in F}G_i * \varprojlim(*_{i\in X}G_i, p_{XY}) : X \subset Y \Subset I \setminus F)$ and show that it is a subgroup of $\varprojlim(*_{i\in X}G_i, p_{XY} : X \subset Y \Subset I)$. Let's denote by H the group $\varprojlim(*_{i\in X}G_i, p_{XY} : X \subset Y \Subset I)$, and for each $F \Subset I$, denote by H_F the group $*_{i\in F}G_i * \varprojlim(*_{i\in X}G_i, p_{XY} : X \subset Y \Subset I \setminus F)$. We are going to show that H_F is canonically injected to H. Fix $F \Subset I$, define the morphism $\varphi_F : H_F \to H$, such that for all $x \in *_{i\in F}G_i, \varphi_F(x) = (p_{X\cap F,F}(x))_{X\Subset I}$, and for all $y = (y_X)_{X\Subset I\setminus F} \in \varprojlim(*_{i\in X}G_i, p_{XY} : X \subset Y \Subset I \setminus F)$, $\varphi_F(y) = (y_{X\setminus F})_{X\Subset I}$. Since the process of reducing an unreduced word in $*_{i\in F}G_i * \varprojlim(*_{i\in X}G_i, p_{XY}) : X \subset Y \Subset I \setminus F)$ does not affect the X-coordinates of its image by φ for any $X \Subset I$, so does not affect its image. Hence, φ_F is well defined morphism. φ_F is clearly injective, which means that we can regard H_F as a subgroup of H, and then the intersection $\bigcap_{F\Subset I} H_F$ has sense and is justified, and it is a subgroup of H.

Let's prove the first part of the proposition. For $X \Subset I$, define $\phi_X : \times_{i\in I}G_i \to *_{i\in X}G_i$, $\phi_X(W) = W_X$ for a word $W \in \times_{i\in I}G_i$, then ϕ_X is well defined morphism. Let $\phi : \times_{i\in I}G_i \to \prod_{X\Subset I} *_{i\in X}G_i$ be the induced morphism by $(\phi_X)_{X\Subset I}$, ϕ is clearly injective. We have for $X \subset Y \Subset I$, $p_{XY} \circ \phi_Y = \phi_X$, which implies that $Im(\phi) \subset \varprojlim(*_{i\in X}G_i, p_{XY} : X \subset Y \Subset I)$. Given $W \in \times_{i\in I}G_i$ reduced, and $F \Subset I$, let $\{\alpha_1 < \alpha_2 < \cdots < \alpha_n\} = W^{-1}(\cup_{i\in F}G_i)$. We can partition the previous set into $\{\alpha_1 < \alpha_2 < \cdots < \alpha_{n_1}\}$, $\{\alpha_{n_1+1} < \cdots < \alpha_{n_2}\}, \cdots \{\alpha_{n_{k-1}+1} < \cdots < \alpha_{n_k}\}$ such any two consecutive elements of each of those subsets are neighboring elements in $\overline{W}$. Let $x_j = W(\alpha_{n_{j-1}+1})W(\alpha_{n_{j-1}+2})\cdots W(\alpha_{n_j}) \in *_{i\in F}G_i$ for $j \in \{1,\ldots,k\}$ and $n_0 = 0$. For each $X \Subset I \setminus F$ let $y_{0X} = W_{X\cap]\infty,\alpha_1[}$ and $y_{jX} = W_{X\cap]\alpha_{n_j},\alpha_{n_j+1}[}$ for $j \in \{1,\ldots,k-1\}$ and $y_{kX} = W_{X\cap]\alpha_k,\infty[}$. We have $y_j := (y_{jX})_{X\Subset I\setminus F}$ belongs to $\varprojlim(*_{i\in X}G_i, p_{XY} : X \subset Y \Subset I \setminus F$ (since $p_{XY} \circ \phi_Y = \phi_X$), and for each $X \Subset I$, $W_X = y_{0X\setminus F}p_{X\cap FF}(x_1)y_{1X\setminus F}p_{X\cap FF}(x_2)\cdots p_{X\cap FF}(x_k)y_{kX\setminus F}$, (note that y_0 and y_k can be zeros and others not). Hence $\phi(W) \in \bigcap_{F\Subset I} *_{i\in F}G_i * \varprojlim(*_{i\in X}G_i, p_{XY} : X \subset Y \Subset I \setminus F)$.

Inversely, given $x \in \bigcap_{F\Subset I} *_{i\in F}G_i * \varprojlim(*_{i\in X}G_i, p_{XY} : X \subset Y \Subset I \setminus F)$. For each $i \in I$, let V_i be the reduced word corresponding as a member of $G_i * \varprojlim(*_{i\in X}G_i, p_{XY} : X \subset Y \Subset I \setminus i)$. Let $g(i,1),\cdots,g(i,k_i)$ be the sequence of letters in G_i appearing in V_i in this order. Let $\overline{W} = \{(i,1),\cdots,(i,k_i) : i \in I\}$. Consider the reduced word $V_{i,j}$ corresponding to x as a member of $G_i * G_j * \varprojlim(*_{i\in X}G_i, p_{XY} : X \subset Y \Subset I \setminus i)$, then using the first paragraph of the proof one can see easily that $g(i,1),\cdots,g(i,k_i)$ and $g(j,1),\cdots,g(j,k_j)$ are appearing in $W_{i,j}$. Define $(i,p) < (j,q)$ if $g(i,p)$ is left of $g(j,q)$, then obviously $\leq$ (i.e $<$ *or* $=$) is a total order in $\overline{W}$. Define W, as $W(i,p) = g(i,p)$, then $W \in \mathcal{W}(G_i : i \in I)$, and $\phi(W) = x$, this completes the proof.

$\square$

3 The Hawaiian Earring Group

In all this part all topological spaces are supposed Hausdorff. The aim of this section is to calculate the fundamental group of the Hawaiian earring space, which we call it the Hawaiian earring group.

A loop is said to be nulhomotopic if it is homotopic to the constant loop.

Definition 3.1. *Given $(X, x_i)_{i\in I}$ a family of pointed spaces, the topological space denoted by $(\tilde{\bigvee}_{i\in I}(X_i, x_i), x^*)$ or just by $\tilde{\bigvee}_{i\in I}X_i$ is the set $\coprod_{i\in I} X_i/\sim$ where $\sim$ is the identification of all $x_i(= x^*)$, equiped with the following topology : a subset U of $\coprod_{i\in I} X_i/\sim$ that doesn't contain x^* is open if for each i, $U \cap X_i$ is open on X_i, and and if $x^* \in U$, U is open if for each i, $U \cap X_i$ is open on X_i, and for almost all $i \in I$, $U \cap X_i = X_i$.*

Example 3.2. *We see obviously that $\mathbb{H}$ is homeomorphic to $\tilde{\bigvee}_{n\in\mathbb{N}}S^1$.*

Before stating the main theorem, let's prove a lemma and provide a definition to aid in the proof of the theorem.

Lemma 3.3. *Let X be locally simply connected at x and has countable neighborhood basis at x. Let f be a loop in $((X, x)\vee(Y, y), x^*)$, then $f^{-1}(X\setminus\{x\})$ is a countable disjoint union of open subintervals of I. Suppose $f^{-1}(X \setminus \{x\}) = \bigsqcup_{n\in\mathbb{N}}]a_n, b_n[$. Then $f(a_n) = f(b_n) = x^*$ for all $n \in \mathbb{N}$. And there exists a continuous map $H : [0, 1] \times [0, 1] \to (X, x) \vee (Y, y)$ with the following:*

1. *$H(1, t) = f(t)$ for $t \in [0, 1]$;*
2. *$H(s, 0) = H(s, 1) = H(s, a_n) = H(s, b_n) = x$ for $s \in [0, 1]$ and $n \in \mathbb{N}$;*
3. *$H(s, t) \in X$ for $s \in [0, 1]$ and $t \in \bigcup_{n\in\mathbb{N}}[a_n, b_n]$;*
4. *$H(0, t) = x$ for $t \in \bigcup\{[a_n, b_n] : f|_{[a_n,b_n]}$ is nulhomotopic in $X\}$.*
5. *$H(s, t) = f(t)$ for $s \in [0, 1]$ and $t \notin \{]a_n, b_n[: f|_{[a_n,b_n]}$ is nulhomotopic in $X\}$.*

Proof. Let $X = V_1 \supset V_2 \supset V_3 \supset \cdots$ be a basis of neighborhoods of x, for each k there is an open neighborhood $U_k \subset V_k$ of x simply connected, and we can suppose for each k $U_{k+1} \subset U_k$, so $(U_k)_{k\in\mathbb{N}}$ is a basis of neighborhoods of x in X. Pose $U_0 = X$.

we are working in Hausdorff spaces, so $X\setminus\{x\}$ is open subset of $X\vee Y$, then $f^{-1}(X\setminus\{x\})$ is open subset of the interval $I = [0, 1]$, so it is a countable disjoint union of open subintervals of I. Putting $S_n =]a_n, b_n[$, by hypothesis $f^{-1}(X \setminus \{x\}) = \sqcup_{n\in\mathbb{N}}S_n$, where $(S_n)_{n\in\mathbb{N}}$ are disjoint open subintervals of I. The image by f of the endpoints of each S_n can not be in $X \setminus \{x\}$ since S_n are disjoint, and can not be in $Y \setminus \{y\}$ by continuity of f, so it is x^*. Similarly $f^{-1}(Y \setminus \{y\}) = \sqcup_{n\in\mathbb{N}}L_n$, where $(L_n)_{n\in\mathbb{N}}$ are disjoint open subintervals of I, and the image by f of the endpoints of each L_n is x^*.

For each open neighborhood V of x in X, almost all $n \in \mathbb{N}$, $f(S_n) \subset V$. Indeed, suppose not, then there is sequence of positive integers $(n_k)_{k\in N}$ such that for each k, there is $x_k \in S_{n_k}$ and $f(x_k) \notin V$. By compactness of I, (x_k) admits a convergent subsequence $(x_{k_m})_{m\in\mathbb{N}}$, let w be its limit. Then both of $(a_{n_{k_m}})$ and $(b_{n_{k_m}})$ converge to w, which implies that f is not continuous on w.

For each $n \in \mathbb{N}$, let m_n be the greatest integer such that $f(S_n) \subset U_{m_n}$. If $m_n \neq 0$, then U_{m_n} is simply connected and $f|_{\overline{S_n}} : \overline{S_n} = [a_n, b_n] \to U_{m_n}$ is a loop. We then fix a homotopy $H_n : I \times [a_n, b_n] \to U_{m_n}$, such that $H_n(0,t) = x, H_n(1,t) = f(t), H_n(s, a_n) = H_n(s, b_n) = x$ for all t and s. And let $n_1, n_2, \ldots, n_l, \ldots, n_k$ be all the positive integers that verifies $m_n = 0$, and $\{n_{l+1}, \ldots, n_k\} = \{n \in \{n_1, \ldots, n_k\} : f|_{[a_n,b_n]}$ is nulhomohtopic in $X\}$, for each $j \in \{l+1, \ldots k\}$, we fix then a homotopy $H_{n_j} : I \times [a_{n_j}, b_{n_j}] \to X$, $H_{n_j}(0,t) = x, H_{n_j}(1,t) = f(t), H_{n_j}(s, a_{n_j}) = H_{n_j}(s, b_{n_j}) = x$ for all t and s.

Define $H : I \times I \to X \vee Y$, as, $H(s,t) = f(t)$ if $t \notin \cup\{S_n : n \in \mathbb{N} \setminus \{n_1, n_2, \ldots, n_l\}\}$, $H(s,t) = H_n(s,t)$ if $t \in S_n$ and $n \notin \{n_1, \ldots, n_l\}$, for all $s \in I$. Let's prove that H is continuous. H is trivially continuous on each point (s,t) verifies $f(t) \neq x^*$.

Fix $(s,t) \in I \times I$ that verifies $f(t) = x^*$, and an open neighborhood V of x^* in $X \vee Y$, we will prove by distinguishing cases that there is some $\epsilon > 0$ such that $H(]s-\epsilon, s+\epsilon[\times[t, t+\epsilon[\subset V$.

1st case There is $\epsilon > 0$, such that $f(]t, t+\epsilon[) \subset X \setminus \{x\}$, in other words $t = a_n$ for some n. Suppose firstly that $n \in \{n_1, \ldots, n_l\}$. f is continuous so there is $\delta > 0$ such that , $f([t, t+\delta[) \subset V$, then $H(I \times [t, t+\min(\epsilon,\delta)[) = f([t, t+\min(\epsilon,\delta)[) \subset V$. Suppose now that n is different from $n_1, \ldots, n_l$. $V \cap U_{m_n}$ is an open neighborhood of x in U_{m_n}, so there is $\delta > 0$ such that $H_n(]s-\delta, s+\delta[\times[t, t+\delta[) \subset V$, hence $H(]s-\delta, s+\delta[\times[t, t+\delta[) = H_n(]s-\delta, s+\delta[\times[t, t+\delta[) \subset V$.

2nd case There is $\epsilon > 0$, such that $f(]t, t+\epsilon[) \subset Y$. f is continuous so there is $\delta > 0$ such that , $f([t, t+\delta[) \subset V$, then $H(I \times [t, t+\min(\epsilon,\delta)[) = f([t, t+\min(\epsilon,\delta)[) \subset V$.

3rd case There is a subsequence $(S_{n_k})_{k\in\mathbb{N}}$ of the intervals (S_n) that converges to t from the right, let $m > 0$ such that $U_m \subset V \cap X$. Since there is just a finite number of S_n such that $f(S_n) \not\subset U_m$, we can suppose that for each k $f(S_{n_k}) \subset U_m$, then for each k $U_{n_k} \subset U_m$, hence for each k, $H(I \times [a_{n_k}, b_{n_k}]) = H_{n_k}(I \times [a_{n_k}, b_{n_k}]) \subset U_m$. Suppose in the first subcase that there is also a subsequence $(L_{n_k})_{k\in\mathbb{N}}$ of the intervals (L_n) that converges to from the right, $V \cap Y$ is open neighborhood of y, for the same argument as (S_n), almost for all n, $f(L_n) \subset V \cap Y$, so we can suppose that for each k $f(L_{n_k}) \subset V \cap Y$, hence for each k $H(I \times \overline{L_{n_k}}) = f(\overline{L_{n_k}}) \subset V \cap Y$.we deduce then the existence of $\epsilon > 0$ that verifies $H(I \times [t, t+\epsilon[) \subset V$. Suppose now the nonexistence of such subsequence of (L_n), then we deduce directly the existence of $\epsilon > 0$ that verifies $H(I \times [t, t+\epsilon[) \subset V$. Similarly if there is such a subsequence of (L_n) and there is not such a subsequence of (S_n).

By a similar argument there is some $\delta > 0$ such that $H(]s-\delta, s+\delta[\times]t-\delta, t] \subset V$. This completes the proof of the continuity of H.

$\square$

Definition 3.4. *A loop f in (X, x) is said to be proper, if f satisfies the following: Let $(]a_n, b_n[)_{n\in M}$ be pairwise disjoint open intervals (of course such M must be countable) such that $\bigcup_{n\in M}]a_n, b_n[= f^{-1}(X \setminus \{f(0)\})$. Then, if $f|_{[a_n,b_n]}$ is homotopic to the constant loop, $f|_{[a_n,b_n]}$ itself is constant.*

Theorem 3.5. ***H.B.Griffiths [3], J.W.Morgan and I.morrisin[4]***

Given $(X_i, x_i)_{i\in I}$ be family of pointed spaces, suppose for each $i \in I$ X_i is locally simply connected at x, and has a countable neighborhood basis at x_i. Then,

$$\pi_1(\tilde{\bigvee}_{i\in I}(X_i, x_i), x^*) \cong \times^{\sigma}_{i\in I}\pi_1(X_i, x_i)$$

.

Proof of the theorem. If we do not mention the domain of a loop, it will be $[0, 1]$.

Let f be a loop in $(\tilde{\bigvee}_{i\in I}(X_i, x_i), x^*)$. $\tilde{\bigvee}_{i\in I}(X_i, x_i), x^*)\setminus\{x^*\}$ is open, so $f^{-1}(\tilde{\bigvee}_{i\in I}(X_i, x_i), x^*)\setminus \{x^*\})$ is open, and then it is a disjoint union of countable open subintervals $(]a_n, b_n[)_{n\in M}$ of $[0, 1]$. Then $f(a_n) = f(b_n) = x^*$ for each $n \in M$. For each n $f(]a_n, b_n[)$ is connected, so $f(]a_n, b_n[) \subset X_i$ for some i. Indeed, fix i such that $f(]a_n, b_n[) \cap X_i \neq \emptyset$, $f(]a_n, b_n[) = (f(]a_n, b_n[) \cap (X_i \setminus \{x_i\})) \sqcup (f(]a_n, b_n[) \cap \sqcup_{j\neq i}(X_j \setminus \{x_j\}))$. Both of $X_i \setminus \{x_i\}$ and $\sqcup_{j\neq i}(X_j \setminus \{x_j\})$ are open in $(\tilde{\bigvee}_{i\in I}(X_i, x_i), x^*)$. Hence, $f(]a_n, b_n[) \cap \sqcup_{j\neq i}(X_j \setminus \{x_j\}) = \emptyset$.

Given $i \in I$, X_i is locally simply connected at x, so there is a simply connected open neighborhood V_i of x in X_i. Almost all loops $f([a_n, b_n])$ lying in X_i verifies $f([a_n, b_n]) \subset V_i$. Indeed, suppose not, then there is sequence of elements of M $(n_k)_{k\in N}$ such that for each k, there is $x_k \in]a_{n_k}, b_{n_k}[$ and $f(x_k) \notin V_i$ and $f([a_{n_k}, b_{n_k}]) \subset X_i$. By compactness of I, (x_k) admits a convergent subsequence $(x_{k_m})_{m\in\mathbb{N}}$, let w be its limit. Then both of $(a_{n_{k_m}})_{m\in\mathbb{N}}$ and $(b_{n_{k_m}})_{m\in\mathbb{N}}$ converge to w, which implies that f is not continuous on w. Hence, for each i almost all loops $f_{|[a_n,b_n]}$ lies in X_i are homotopic to the constant map. Define $\overline{W^f} = \{n \in M : f_{|[a_n,b_n]}$ is not nulhomotopic in X_i where $f_{|[a_n,b_n]}$lies in $X_i\}$ equipped with the order of $(a_n)_{n\in M}$, and $W^f(n) = [f_{|[a_n,b_n]}]$ as an element of $\pi_1(X_i, x_i)$ if $f([a_n, b_n]) \subset X_i$. Then $W^f \in \mathcal{W}(\pi_1(X_i, x_i) : i \in I)$.

Fix $F \Subset I$, the subspace topology $\{x^*\} \cup_{i\in F} X_i \setminus \{x_i\}$ of $\tilde{\bigvee}_{i\in I}X_i$ is equal to the topological space $\bigvee_{i\in F} X_i$. And the canonical map $\pi_F : \tilde{\bigvee}_{i\in I}X_i \to \bigvee_{i\in F} X_i$ that conserves the X_i for $i \in F$ and sends the other points to x^* is continuous. And $[\pi_F \circ f] = (W^f)_F$. Hence if H is a homotopy between f and the constant map, then $\pi_F \circ H$ is a homotopy between $\pi_F \circ f$ and the constant map. So, if $[f] = 0$ then $(W^f)_F = e$ for each $F \Subset I$, and then $[W^f] = e$.

Define $\psi : \pi_1(\tilde{\bigvee}_{i\in I}(X_i, x_i), x^*) \to \times^{\sigma}_{i\in I}\pi_1(X_i, x_i)$, as $\psi([f]) = [W^f]$ for each loop f in $(\tilde{\bigvee}_{i\in I}(X_i, x_i), x^*)$. We have trivially for two loops f, g in $\tilde{\bigvee}_{i\in I}(X_i, x_i), x^*)$, $W^{f.g} = W^f W^g$ and $W^{\overline{g}} = (W^g)^{-1}$, where $\overline{g}$ is the inverse path of g. Suppose f and g are homotopic, $[f.\overline{g}] = 0$ so, $[W^f][W^{\overline{g}}] = [W^f W^{\overline{g}}] = [W^{f.\overline{g}}] = e$, then $[W^f] = [(W^{\overline{g}})^{-1}] = [W^g]$. Hence, ψ is well defined homomorphism.

Let's show that ψ is surjective. Fix a σ-word $W \in \mathcal{W}_\sigma(\pi_1(X_i, x_i) : i \in I)$. If $\overline{W}$ is finite it is trivial that it is has a preimage. Suppose that $\overline{W}$ is infinite and reduced, we will construct a loop f such that $W^f = W$. Fix $\alpha_1 \in \overline{W}$, and define f on $[1/3, 2/3]$ such that $f|_{[1/3,2/3]} \in W(\alpha_1)$. If there is $\alpha_1 > \alpha_{2,1} \in \overline{W}$, then define f on $[1/9, 2/9]$ such that $f|_{[1/9,2/9]} \in W(\alpha_{2,1})$. If there is $\alpha_1 < \alpha_{2,2} \in \overline{W}$, then define f on $[7/9, 8/9]$ such that $f|_{[7/9,8/9]} \in W(\alpha_{2,2})$. If there is $\alpha_{2,1} > \alpha_{3,1} \in \overline{W}$, then define f on $[1/27, 2/27]$ such that $f|_{[1/27,2/27]} \in W(\alpha_{3,1})$. If there is $\alpha_{3,2} \in \overline{W}$ such that $\alpha_1 > \alpha_{3,2} > \alpha_{2,1}$, then define f on $[7/27, 8/27]$ such that $f|_{[7/27,8/27]} \in W(\alpha_{3,2})$. We continue this procedure infinitely many times, and define f on points that are not defined in the previous procedure to be x^*. We could actually make things to look more rigorously by defining f as limit of a sequence of functions. Let's show that f is continuous.

The continuity of f is trivial on $\bigcup_{n\in\mathbb{N}}\bigcup_{0\le k\le 3^{n-1}-1}]\frac{3k+1}{3^n}, \frac{3k+2}{3^n}[$. Let $t \in [0,1] \setminus \bigcup_{n\in\mathbb{N}}\bigcup_{0\le k\le 3^{n-1}-1}]\frac{3k+1}{3^n}, \frac{3k+2}{3^n}[$, then $f(t) = x^*$, let V be an open neighborhood of x^*, we will prove that there is $\epsilon > 0$ such that $f([t, t+\epsilon[) \subset V$. This holds trivially if $t = \frac{3k+1}{3^n}$, or if there is $\epsilon > 0$ such that $f([t, t+\epsilon[) = \{x^*\}$. Suppose this is not the case, then there exists a sequence of the intervals $\{[\frac{3k+1}{3^n}, \frac{3k+2}{3^n}] : n \in \mathbb{N}, 0 \le k \le 3^{n-1} - 1\}$ that converges to t from the right, and f restricted on each interval of the sequence is not nulhomptopic. We have $V \cap X_i = X_i$ for almost all $i \in I$, and for each $i \in I$, $W^{-1}(\pi_1(X_i, x_i))$ is finite, then we can suppose that the image of each interval of the sequence is included in V, then we deduce the existence of $\epsilon > 0$ such that $f([t, t+\epsilon[) \subset V$. Similarly we show the existence of $\delta > 0$ such that $f(]t-\delta, t]) \subset V$. Hence, f is continous. And we have obviously $W^f = W$.

To prove the injectivity of ψ, we need the following lemma:

Lemma 3.6. *Any loop f in $\tilde{\bigvee}_{i\in I}(X_i, x_i), x^*)$ is homotopic to some proper loop.*

proof of the lemma: Given a loop f in $\tilde{\bigvee}_{i\in I}(X_i, x_i), x^*)$, we have proven that the image of f is included in $\tilde{\bigvee}_{i\in C}(X_i, x_i), x^*)$ for some countable $C \subset I$, if C is finite, there is nothing to prove, just applying lemma 3.2 finitely many times. suppose C is infinite, let's enumerate its elements $C = \{i_1, i_2, \ldots, \}$.We remark that for each $i \in I$ the topological space $X_i \vee \tilde{\bigvee}_{i\neq j\in I}X_j$ is equal to $\tilde{\bigvee}_{i\in I}X_i$. We construct $f_n, H_n, (n \in \mathbb{N})$

by induction, $f_1 = f$, applying lemma 3.2 on f_1, with a small variation in the domain we get a map $H_1 : [1/2, 1] \times [0, 1] \to X_{i_1} \vee \tilde{\bigvee}_{i_1 \neq j \in I} X_j$ that verifies the conditions given in same lemma. Suppose we have constructed $f_n : [0, 1] \to \tilde{\bigvee}_{i \in I} X_i$ and $H_n : [1/(n+1), 1/n] \times [0, 1] \to X_{i_n} \vee \tilde{\bigvee}_{i_n \neq j \in I} X_j$, we take $f_{n+1} = H_n(\frac{1}{n+1}, \bullet)$, applying lemma 3.2 on f_{n+1}, we get a map $H_{n+1} : [\frac{1}{n+2}, \frac{1}{n+1}] \times [0, 1] \to X_{i_{n+1}} \vee \tilde{\bigvee}_{i_{n+1} \neq j \in I} X_j$ that verifies the conditions.

For each i The map $\pi_{\{i\}} : \tilde{\bigvee}_{i \in I} X_i \to X_i$ is a retraction, then by Proposition 1.12, the homomorpism $i_* : \pi_1(X_i, x_i) \to \pi_1(\tilde{\bigvee}_{i \in I} X_i, x^*)$ induced by the inclusion is injective. Then, a loop that lies in X_i on x_i is nulhomotopic in $\tilde{\bigvee}_{i \in I} X_i$, if and only if it is nulhomotopic in X_i. Define $g : [0, 1] \to \tilde{\bigvee}_{i \in I} X_i$ defined as, $g(t) = x^*$ if $t \in]a_n, b_n[$, $n \in M$ for which $f|_{[a_n, b_n]}$ is nulhomotopic, $g(t) = f(t)$ otherwise. g is clearly continuous on $\cup_{n \in M}]a_n, b_n[$, let $t \in [0, 1] \setminus \cup_{n \in M}]a_n, b_n[$ then $g(t) = f(t) = x^*$, let V be a neighborhood of x^*, then there is $\epsilon > 0$, $f(]t - \epsilon, t + \epsilon[) \subset V$, then $g(]t - \epsilon, t + \epsilon[) \subset f(]t - \epsilon, t + \epsilon[) \subset V$, so g is continuous. Hence, g is a proper loop.

Define $H : [0, 1] \times [0, 1] \to \tilde{\bigvee}_{i \in I} X_i$ as, $H(s, t) = H_n(s, t)$ if $s \in [\frac{1}{n+1}, \frac{1}{n}]$, and $H(0, t) = g(t)$. H is trivially continuous on $]0, 1] \times [0, 1]$. Given $t \in]a_n, b_n[, n \in M$ for which $f|_{[a_n, b_n]}$ is nulhnmotopic. Let $m \in \mathbb{N}$ such that $f([a_n, b_n]) \in X_{i_m}$, then by the properties 4 and 5 of lemma 3.2, we get for all $k > m$ for all $s \in [\frac{1}{k+1}, \frac{1}{k}], t \in [a_n, b_n]$ $H_k(s, t) = f_{m+1}(t) = x^*$, then we deduce that H is continuous on $\{0\} \times]a_n, b_n[$, for all $n \in M$ for which $f|_{[a_n, b_n]}$ is nulhnmotopic. Given $t \in]a_n, b_n[, n \in M$ for which $f|_{[a_n, b_n]}$ is not nulhnmotopic. Then, for each $k \in \mathbb{N}$, for each $s \in [\frac{1}{k+1}, \frac{1}{k}]$, $H_k(s, t) = f(t)$. Hence H is continuous on $\{0\} \times]a_n, b_n[$, for all $n \in M$ for which $f|_{[a_n, b_n]}$ is not nulhnmotopic.

Given $t \in [0, 1] \setminus \cup_{n \in M}]a_n, b_n[$, and V an open neighborhood of $H(0, t) = x^*$, we are going to prove by studying cases that there is $\epsilon > 0$ such that $H([0, \epsilon[\times [t, t + \epsilon[) \subset V$. If $t = a_n$ for some $n \in M$, it is trivial result based on the foregoing paragraph. If there is $\epsilon > 0$ such that $f([t, t + \epsilon[) = \{x^*\}$, then $H([0, 1] \times [t, t + \epsilon[) = \{x^*\}$,so there nothing to prove. Suppose now that none of the previous cases hold, then there is a sequence $(n_k)_{k \in \mathbb{N}}$ over M, such that $(]a_{n_k}, b_{n_k}[)_{k \in \mathbb{N}}$ converges to t from the right. We know that for almost all $i \in I$, $V \cap X_i = X_i$, and since for each i almost all loops that lies in X_i are nulhomotopic, then we can suppose that for each k: if $f(]a_{n_k}, b_{n_k}[) \subset X_i$ and $V \cap X_i \neq X_i$ then $f|_{[a_{n_k}, b_{n_k}]}$ is nulhnmotopic. Let $m \in \mathbb{N}$ such that for all $k \geq m$, $V \cap X_{i_k} = X_{i_k}$. Then, for each $l \geq m$, for each $k \in \mathbb{N}$, if $f(]a_{n_k}, b_{n_k}[) \subset X_i$ and $V \cap X_i \neq X_i$ then $H_l([\frac{1}{l+1}, \frac{1}{l}] \times]a_{n_k}, b_{n_k}[) = \{x^*\}$, and if not then $H_l([\frac{1}{l+1}, \frac{1}{l}] \times]a_{n_k}, b_{n_k}[) \subset X_i$ (just verify the cases when l is greater or equal or less than the index of i in C). Hence we deduce the desired result. Similarly we prove the existence of $\delta > 0$ such that $H([0, \delta[\times]t - \delta, t]) \subset V$, This achieves the proof of continuity of H.

We conclude finally that f is homotopic to the proper loop g. □

Let's prove now that ψ is injective, we will present the proof of Eda [1] with more details. Let's denote $\pi_1(X_i, x_i)$ by G_i. Let f be a loop in $(\tilde{\bigvee}_{i\in I}(X_i, x_i), x^*)$ with $[W^f] = e$. Since there exists a countable subset C of I such that $\mathrm{Im}(f) \subset \tilde{\bigvee}_{i\in C} X_i$, it suffices to deal the case $I = \mathbb{N}$. By Lemma 3.6, we may assume that f is a proper loop. Now, we construct a homotopy H from f to the constant loop. In the k-th step, we define H on subrectangles of $[0, 1] \times [0, 1]$ which makes loops in (X_k, x^*) homotopic to the constant loop expecting loops in $\tilde{\bigvee}_{n>k} X_n$ will be made homotopic to the constant loop in a suitable way in future.

Step 1 Let $H(t, 1) = f(t)$ and $H(t, 0) = x^*$ for $0 \le t \le 1$. Let $W^f = W_1 \cdots W_{n_1}$, where $W_i \in \mathcal{W}(G_1)$ or $W_i \in \mathcal{W}(G_n : n \ge 2)$ for $1 \le i \le n_1$ and $W_i \in \mathcal{W}(G_1)$ if and only if $W_{i+1} \in \mathcal{W}(G_j : j \ge 2)$ for $1 \le i \le n_1 - 1$.

Substep 1: We can correspond a closed interval I_i to each W_i so that $W_i = W^{f|_{I_i}}$ for $1 \le i \le n_1$, $\bigcup_{i=1}^{n_1} I_i = [0, 1]$, and the right end of I_i is the left end of I_{i+1} for $1 \le i \le n_1 - 1$. We claim that $W_i = e$ for some $1 \le i \le n_1$. Suppose not. There exists $F \Subset \mathbb{N}$ such that $(W_i)_F \ne e$ for every $1 \le i \le n_1$. Then, $(W^f)_F \ne e$, which is a contradiction. We choose one W_i with $W_i = e$. Let $H(s, t) = f(s)$ for $(s, t) \in \bigcup_{j\ne i} I_j \times [1/2, 1]$.

In case $W_i \in \mathcal{W}(G_1)$, $f|_{I_i}$ is homotopic to the constant loop in X_i. Let $H|_{I_i\times[1/2,1]}$ be a continuous map such that $H(s, 1/2) = x^*$ for $s \in I_i$, and $H(s, t) = x^*$ for $s \in \partial I_i$ and $t \in [1/2, 1]$. In case $W_i \in \mathcal{W}(G_n : n \ge 2)$, we do not define H on $(\mathrm{int} I_i)\times]1/2, 1[$ in this step, but we let $H(s, 1/2) = x^*$ for $s \in I_i$. Next, we reform the word W^f to $W_1 \cdots V \cdots W_{n_1}$ by eliminating W_i, where $V = W_{i-1}W_{i+1}$. Then, $W_1 \cdots V \cdots W_{n_1} = e$ and members of $\mathcal{W}(G_1)$ and $\mathcal{W}(G_n : n \ge 2)$ are neighboring in $W_1, \cdots, V, \cdots, W_{n_1}$.

Substep k + 1: In the substep k, $H(s, 1/2^k)$ $(s \in [0, 1])$ have been defined and there is a corresponding word reformed from W^f. By the same reasoning as in Substep 1, one of the words equals e as a member of the group, of course. We perform the work as in Substep 1. The substeps would finish in at most n_1-steps. If they finish in the k-step, then $H(t, 1/2^k)$ $(0 \le t \le 1)$ have been defined and equal to x^*. Let $H([0, 1] \times [0, 1/2^k]) = x^*$.

Step k After the $(k - 1)$-step, there possibly exist finitely many sub-rectangles of $[0, 1] \times [0, 1]$ on which H has not been defined. Their forms are $[a, b]\times]\sum_{i=1}^{m-1} s_i/2^i + 1/2^m, \sum_{i=1}^{m-1} s_i/2^i + 1/2^{m-1}[$, where $s_i = 0$ or 1 and $m \le \sum_{i=1}^{k} n_i$. H has been defined on the upper side of a rectangle and it corresponds to a word in $\mathcal{W}(G_n : n \ge k)$, then it has values in $\tilde{\bigvee}_{n\ge k} X_n$ since f is a proper loop. H maps the lower side to x^*. In each rectangle, we work as in Step 1, as if the rectangle were $[0, 1] \times [0, 1]$. Note that the values of H which we define in this step are in $\tilde{\bigvee}_{n\ge k} X_n$.

Let $H(s, t) = x^*$, if $H(s, t)$ has not been defined in any step. Now, let's show the continuity of H. Given $u \in [0, 1] \times [0, 1]$, suppose in the first case that there is an injective sequence of subrectanges resulting from the construction $(L_m)_{m\in\mathbb{N}}$ such that $\lim_{m\to 0} d(u, L_m) = 0$. Then, $H(u) = x^*$. Given V an open neighborhood of x^*. There is $k \in \mathbb{N}$ such that $\tilde{\bigvee}_{n\geq k} X_n \subset V$, and there is $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, $H(L_m) \subset \tilde{\bigvee}_{n\geq k} X_n$. Then, we deduce the continuity of H on u. In the other case H is obviously continuous on u. Hence H is continuous and the proof of the theorem is complete.

□

Corollary 3.7. $\pi_1(\mathbb{H}) \cong \times^{\sigma}_{n\in\mathbb{N}}\mathbb{Z} \cong \times_{n\in\mathbb{N}}\mathbb{Z}$.

Corollary 3.8. $\pi_1(H)$ *is isomorphic to*

$$\bigcap_{F\Subset\mathbb{N}} *_{n\in F}\mathbb{Z} * \varprojlim(*_{n\in X}\mathbb{Z}, p_{XY} : X \subset Y \Subset \mathbb{N} \setminus F)$$

which is a subgroup of

$$\varprojlim(*_{n\in X}\mathbb{Z}, p_{XY} : X \subset Y \Subset \mathbb{N}).$$

Hence $\pi_1(\mathbb{H})$ *embeds in an inverse limit of free groups.*

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