Probabilités

Lois gaussiennes

Rappel : $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$

$$\left[\left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2 = \int_{\mathbb{R}^2} e^{-x^2-y^2} dx\,dy \underset{\substack{x = r\cos\theta \\ y = r\sin\theta}}{=} \int_0^{\infty}\int_{-\pi}^{\pi} e^{-r^2}\, r\,dr\,d\theta = 2\pi\left[-\frac{e^{-r^2}}{2}\right]_0^{\infty} = \pi\right]$$

Définition : Une v.a. réelle X suit une loi gaussienne $\mathcal{N}(m, \sigma^2)$ si

espérance

variance

$$\forall f \in C_b(\mathbb{R}), \quad E(f(X)) = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} f(x)\, e^{-\frac{(x-m)^2}{2\sigma^2}} dx.$$

continue bornée

Remarques

1) $\displaystyle \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx = 1$

2) $$E(X) = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} x\, e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$
$$= \underbrace{\frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} (x-m)\, e^{-\frac{(x-m)^2}{2\sigma^2}} dx}_{0} + \frac{m}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$
$$= m$$

3) $$\mathrm{var}(X) = E(X^2) - (E(X))^2$$
$$= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} x^2 e^{-\frac{(x-m)^2}{2\sigma^2}} dx - m^2$$
$$\underset{y = x-m}{=} \frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} (y+m)^2 e^{-\frac{y^2}{2\sigma^2}} dy - m^2$$
$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} y^2 e^{-\frac{y^2}{2\sigma^2}} dy \underset{\text{ipp}}{=} \frac{\sigma}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{y^2}{2\sigma^2}} dy \underset{t = \frac{y}{\sqrt{2}\sigma}}{=} \frac{\sigma^2}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-t^2} dt = \boxed{\sigma^2}$$

$u = y \quad v' = y e^{-\frac{y^2}{2\sigma^2}}$

$u' = 1 \quad v = -\sigma^2 e^{-\frac{y^2}{2\sigma^2}}$

Définition (fonction caractéristique)

Si X v.a réelle, on pose $\Phi_X(\xi) = E(e^{i\xi X})$

(fonction caractéristique de X)

ex : si $X \sim \mathcal{N}(0,\sigma^2)$, alors $\Phi_X(\xi) = e^{-\frac{\sigma^2\xi^2}{2}}$

En effet :
$$\Phi_X(\xi) = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} e^{i\xi x} e^{-\frac{x^2}{2\sigma^2}}\,dx$$

$$\Phi_X'(\xi) \underset{\text{dérivation sous }\int}{=} \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} ix e^{i\xi x} e^{-\frac{x^2}{2\sigma^2}}\,dx$$

$$\underset{\text{ipp}}{=} -\frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} \xi e^{i\xi x}\sigma^2 e^{-\frac{x^2}{2\sigma^2}}\,dx$$

$u = ie^{i\xi x}$ $\quad v' = xe^{-\frac{x^2}{2\sigma^2}}$

$u' = -\xi e^{i\xi x}$ $\quad v = -\sigma^2 e^{-\frac{x^2}{2\sigma^2}}$

$$= -\sigma^2\xi\,\Phi_X(\xi)$$

$\Phi_X' = -\sigma^2\xi\Phi_X \implies \Phi_X(\xi) = C e^{-\frac{\sigma^2\xi^2}{2}}$ où $C = \Phi_X(0) = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} e^{-\frac{x^2}{2\sigma^2}}\,dx = 1$

$$= e^{-\frac{\sigma^2\xi^2}{2}}.$$

Proposition « Φ_X caractérise X »

Si X_1 va de loi de densité f_1
si X_2 va de loi de densité f_2
alors $\Phi_{X_1} = \Phi_{X_2} \implies \forall \varphi \in C_b(\mathbb{R})$, $\underbrace{\int_{\mathbb{R}} f_1(x)\varphi(x)\,dx}_{E(\varphi(X_1))} = \underbrace{\int_{\mathbb{R}} f_2(x)\varphi(x)\,dx}_{E(\varphi(X_2))}$

démo.

soit f L^1

soit $p_\sigma(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}$ $\quad (\forall \sigma > 0)$

on pose $f_\sigma = p_\sigma * f$

c-à-d : $\forall x$, $f_\sigma(x) = \int_{\mathbb{R}} p_\sigma(x-y) f(y)\,dy$

1) $\forall \varphi \in C_b(\mathbb{R})$, $\lim_{\substack{\sigma\to 0\\ \sigma>0}} \int_{\mathbb{R}} f_\sigma(x)\varphi(x)\,dx = \int_{\mathbb{R}} f(x)\varphi(x)\,dx$.

En effet : $\int_{\mathbb{R}}\int_{\mathbb{R}} p_\sigma(x-y) f(y)\varphi(x)\,dx\,dy = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2\sigma^2}} f(y)\varphi(x)\,dx\,dy$

$$\underset{t=\frac{x-y}{\sigma}}{=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\frac{t^2}{2}} f(y)\, \varphi(y+\sigma t)\, dt\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \underbrace{\left(\int_{\mathbb{R}} e^{-\frac{t^2}{2}} \varphi(y+\sigma t)\, dt \right)} f(y)\, dy$$

$$\forall y\ \forall t \quad e^{-\frac{t^2}{2}} \varphi(y+\sigma t) \underset{\substack{\sigma \to 0 \\ \sigma > 0}}{\longrightarrow} e^{-t^2/2} \varphi(y)$$

$$\xrightarrow[\substack{\sigma \to 0 \\ \sigma > 0}]{\text{(conv. dominée)}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\frac{t^2}{2}} \varphi(y) f(y)\, dt\, dy$$

$$= \int_{\mathbb{R}} f(y) \varphi(y)\, dy.$$

2) on pose $\widehat{p_\sigma}(\xi) = e^{-\frac{\sigma^2 \xi^2}{2}} = \sqrt{2\pi}\, p_{1/\sigma}(\xi)$

$$= E\left(e^{i\xi X_{p_\sigma}}\right)$$

$$= \int_{\mathbb{R}} e^{ix\xi} p_\sigma(x)\, dx.$$

$$\Rightarrow \forall \sigma > 0,\ p_\sigma(\xi) = \frac{1}{\sqrt{2\pi}} \widehat{p_{1/\sigma}}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{it\xi} p_{1/\sigma}(t)\, dt$$

Soit $\varphi \in C_b(\mathbb{R})$, $\int_{\mathbb{R}} f_{1,\sigma}(x) \varphi(x)\, dx = \int_{\mathbb{R}} \int_{\mathbb{R}} p_\sigma(x-y) \varphi(x) f_1(y)\, dx\, dy$

$$\underset{\substack{\sigma \to 0 \\ \sigma > 0}}{\longrightarrow} \int_{\mathbb{R}} f_1(x) \varphi(x)\, dx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{it(x-y)} f_1(y) \varphi(x) p_{1/\sigma}(t)\, dx\, dy\, dt$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \underbrace{\left(\int_{\mathbb{R}} e^{-ity} f_1(y)\, dy \right)}_{\Phi_{X_1}(-t)} \varphi(x) p_{1/\sigma}(t)\, dx\, dt$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi_{X_2}(-t)\, \varphi(x)\, p_{1/\sigma}(t)\, dx\, dt$$

$$= \int_{\mathbb{R}} f_{2,\sigma}(x) \varphi(x)\, dx \underset{\substack{\sigma \to 0 \\ \sigma > 0}}{\longrightarrow} \int_{\mathbb{R}} f_2(x) \varphi(x)\, dx$$

$$\int_{\mathbb{R}} f_1(x) \varphi(x)\, dx = \int_{\mathbb{R}} f_2(x) \varphi(x)\, dx$$

donc $E(\varphi(X_1)) = E(\varphi(X_2))$.

FIN

Prochain cours le jeudi 9 avril