Probabilités

1) Définitions

Un espace de probabilité est un $(\underset{\text{ensemble}}{\Omega}, \overset{\text{tribu}}{\mathcal{A}}, \underset{\text{mesure}}{P})$

tq $P(\Omega) = 1$

Une variable aléatoire (v.a.) est une fonction $X : \Omega \longrightarrow (E, \mu)$ espace mesuré
mesurable

si $E = \mathbb{N}$ v.a. discrète
$E = \mathbb{R}$ v.a. continue

si $X : \Omega \to E$ v.a., la loi de X est

$$\underset{\text{mesurable}}{I \subset E} \longmapsto P(X^{-1}(I)) = P(\{\omega \in \Omega : X(\omega) \in I\})$$

notée parfois $P(X \in I)$

Cas où $E = \mathbb{N}$ Loi de X : $n \longmapsto P(X = n) = P(X^{-1}(\{n\}))$

Cas où $E = \mathbb{R}$ loi de X : $\underset{\text{intervalle}}{I \subset \mathbb{R}} \longmapsto P(X \in I)$

Si $X : \Omega \to \mathbb{N}$ v.a, on note $E(X) = \sum_{n \in \mathbb{N}} n\, P(X = n)$ (si convergence)

Si $X : \Omega \to \mathbb{R}$ v.a, si $\exists f : \mathbb{R} \to \mathbb{R}$,
$\forall I \subset \mathbb{R}$ intervalle, $P(X \in I) = \int_I f$

on note $E(X) = \int_{\mathbb{R}} x f(x)\, dx$ (si convergence)

$E(X)$ espérance de X

Définition: $\underset{\text{variance}}{\mathrm{var}(X)} = E((X - E(X))^2) = E(X^2) - E(X)^2$ (si convergence)
(≥ 0)

Exemples 1) Bernoulli

Soit $0 \leq p \leq 1$, $X : \Omega \longrightarrow \{0, 1\}$

tq $P(X = 0) = 1 - p$, $P(X = 1) = p$

Alors $E(X) = 0 \times P(X=0) + 1 \times P(X=1) = p$

$\mathrm{var}(X) = p - p^2 = p(1-p)$

2) Binomiale $\quad X : \Omega \to \{0, \dots, n\}$

$0 \leq p \leq 1$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad \left(\sum_{k=0}^{n} P(X=k) = 1 \right)$$

$$E(X) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$$

$$= n \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$$

$$= n \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} p^{k+1} (1-p)^{n-1-k}$$

$$= np$$

$$\mathrm{var}(X) = E(X^2) - (np)^2$$

Or $E(X^2) = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k}$

$$= \sum_{k=0}^{n} k(k-1) \binom{n}{k} p^k (1-p)^{n-k} + \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=2}^{n} n(n-1) \binom{n-2}{k-2} p^k (1-p)^{n-2-(k-2)} + np$$

$$= n(n-1) p^2 + np$$

$$\Rightarrow \mathrm{var}(X) = n(n-1)p^2 + np - n^2p^2$$

$$= np(1-p)$$

3) P_λ Poisson

Soit $\lambda > 0$. $\quad P(X=n) = e^{-\lambda} \frac{\lambda^n}{n!} \qquad (n \in \mathbb{N})$

$$\left(\sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} = 1 \right)$$

$$E(X) = \sum_{n=0}^{\infty} n e^{-\lambda} \frac{\lambda^n}{n!} = e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!}$$

$$= \lambda e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = \lambda$$

$$\mathrm{var}(X) = \sum_{n=0}^{\infty} n^2 e^{-\lambda} \frac{\lambda^n}{n!} - \lambda^2$$

$$= \sum_{n=1}^{\infty} \frac{n e^{-\lambda} \lambda^n}{(n-1)!} - \lambda^2 = \sum_{n=1}^{\infty} \frac{(n-1) e^{-\lambda} \lambda^n}{(n-1)!} + \sum_{n=1}^{\infty} \frac{e^{-\lambda} \lambda^n}{(n-1)!} - \lambda^2$$

$$= \sum_{n=2}^{\infty} \frac{e^{-\lambda} \lambda^n}{(n-2)!} + \sum_{n=1}^{\infty} \frac{e^{-\lambda} \lambda^n}{(n-1)!} - \lambda^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda$$

4) Géométrique

Soit $0 < p \leq 1$

$$P(X=n) = p(1-p)^n \qquad \left(\sum_{n=0}^{\infty} p(1-p)^n = p\frac{1}{1-(1-p)} = 1\right)$$

$$E(X) = \sum_{n=0}^{\infty} n p (1-p)^n$$

Or $\displaystyle\sum_{n=0}^{\infty} n x^n = \left(\sum_{n=0}^{\infty} x^n\right)' x = \left(\frac{1}{1-x}\right)' \cdot x = \frac{x}{(1-x)^2}$

$\forall\ 0 \leq x < 1$

donc $\displaystyle E(X) = p\frac{1-p}{(1-(1-p))^2} = \frac{1-p}{p}.$

$$\mathrm{var}(X) = \sum_{n=0}^{\infty} p n^2 (1-p)^n - \left(\frac{1-p}{p}\right)^2$$

Or $\displaystyle\sum_{n=0}^{\infty} n^2 x^n = \sum_{n=0}^{\infty} n(n-1)x^n + \sum_{n=0}^{\infty} n x^n$

$$= x^2\left(\frac{1}{1-x}\right)'' + x\left(\frac{1}{1-x}\right)'$$

$$= \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$$

donc $\displaystyle \mathrm{var}(X) = p\left[\frac{2(1-p)^2}{p^3} + \frac{1-p}{p^2}\right] - \left(\frac{1-p}{p}\right)^2 = \frac{(1-p)^2}{p^2} + \frac{1-p}{p}$

$$= \frac{1-2p+\cancel{p^2}+p-\cancel{p^2}}{p^2}$$

$$= \frac{1-p}{p^2}.$$

5) Loi exponentielle

$$P(X \in I) = \int_I \lambda e^{-\lambda x} 1_{\mathbb{R}_{\geq 0}}(x)\,dx \qquad \left[\int_{\mathbb{R}_{\geq 0}} \lambda e^{-\lambda x}\,dx = 1\right]$$

$$E(X) = \int_{\mathbb{R}_{\geq 0}} x\lambda e^{-\lambda x}\,dx = \left[-xe^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx$$

$u = x \quad v' = \lambda e^{-\lambda x}$

$u' = 1 \quad v = -e^{-\lambda x}$

$$= \frac{1}{\lambda}$$

$$\mathrm{var}(X) = \int_{\mathbb{R}_{\geq 0}} x^2 \lambda e^{-\lambda x}\,dx - \frac{1}{\lambda^2}.$$

Or $\int_0^\infty x^2 \lambda e^{-\lambda x}\, dx = \cancel{\left[-x^2 e^{-\lambda x}\right]_0^\infty} + 2\int_0^\infty x\, e^{-\lambda x}\, dx$

$$= \frac{2}{\lambda^2}$$

$$\mathrm{var}(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}.$$

Pause 5'.