Calcul différentiel

Immersion et submersion

Définitions Soit $f : \underset{\underset{\mathbb{R}^m}{\cap}}{U} \longrightarrow \mathbb{R}^n$ $\mathcal{C}^k$. Soit $a \in U$
ouvert

i) On dit que f est une <u>immersion</u> en a si $df|_a : \mathbb{R}^m \to \mathbb{R}^n$ injective $\Rightarrow m \leq n$

ii) On dit que f est une <u>submersion</u> en a si $df|_a : \mathbb{R}^m \to \mathbb{R}^n$ surjective $\Rightarrow m \geq n$

<u>ex :</u>

$$\begin{aligned} \mathbb{R}^m &\longrightarrow \mathbb{R}^{m+k} \\ x &\longmapsto (\underbrace{x}_{m \text{ variables}}, \underbrace{0}_{k \text{ variables}}) \end{aligned}$$ immersion

$$\begin{aligned} \mathbb{R}^n &\longrightarrow \mathbb{R}^{n-k} \\ (x_1,\dots,x_n) &\longmapsto (x_1,\dots,x_{n-k}) \end{aligned}$$ $(k \geq 0)$ submersion

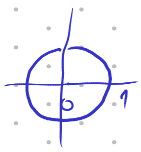
$$\begin{aligned} \mathbb{R}^2 &\xrightarrow{f} \mathbb{R} \\ (x,y) &\longmapsto x^2+y^2-1 \end{aligned}$$

submersion en $(x,y) \neq (0,0)$

[en effet : $df|_{(x,y)} = \begin{pmatrix} 2x & 2y \end{pmatrix} \in M_{12}(\mathbb{R})$]

rang 1 $\forall x,y \neq (0,0)$

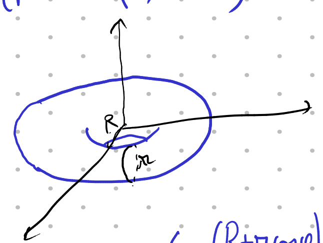
$$\begin{aligned} \mathbb{R} &\xrightarrow{g} \mathbb{R}^2 \\ t &\longmapsto (\cos t, \sin t) \end{aligned}$$



immersion car $(\forall t \in \mathbb{R})$

$dg|_t = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $(\forall t)$ injectif $\mathbb{R} \to \mathbb{R}^2$

$$\begin{aligned} \mathbb{R}^2 &\xrightarrow{h} \mathbb{R}^3 \\ (\theta,\varphi) &\longmapsto \big((R+r\cos\varphi)\cos\theta,\ (R+r\cos\varphi)\sin\theta,\ r\sin\varphi\big) \end{aligned}$$ <u>$0<r<R$</u>



immersion en (θ,φ) $(\forall (\theta,\varphi) \in \mathbb{R}^2)$

en effet : $dh|_{\theta,\varphi} = \mathrm{Jac}(h)|_{\theta,\varphi} = \begin{pmatrix} -(R+r\cos\varphi)\sin\theta & -r\sin\varphi\cos\theta \\ (R+r\cos\varphi)\cos\theta & -r\sin\varphi\sin\theta \\ 0 & r\cos\varphi \end{pmatrix}$

injectif

Théorème a) Soit $f : \underset{\underset{\mathbb{R}^m}{\cap \text{ ouvert}}}{U} \longrightarrow \mathbb{R}^n$ immersion $\mathcal{C}^k$ en $a \in U$

Alors il existe $f(a) \in V \subset \mathbb{R}^n$ ouvert et $\psi : V \to \psi(V) \subset \mathbb{R}^n$ $\mathcal{C}^k$ difféomorphisme tq :

$$\forall \underset{(x_1,\dots,x_m)}{\overset{\shortparallel}{x}} \in f^{-1}(V) \quad , \quad \psi(f(x)) = (x_1,\dots,x_m, 0\cdots 0)$$

b) Soit $f : \underset{\underset{\mathbb{R}^m}{\cap \text{ ouvert}}}{U} \longrightarrow \mathbb{R}^n$ submersion $\mathcal{C}^k$ en $a \in U$ $(m \geq n)$

alors il existe $\Phi : V \to \underset{\underset{a}{\shortparallel}}{\Phi(V)} \subset U$ $\mathcal{C}^k$ difféomorphisme tq

$$\forall x = (x_1,\dots,x_m) \in V, \quad f(\Phi(x_1,\dots,x_m)) = (x_1,\dots,x_n)$$

<u>ex:</u> 1°) $\begin{array}{l} \mathbb{R} \to \mathbb{R}^2 \\ t \mapsto (\cos t, \sin t) \end{array}$ $\quad \psi(x,y) = (\arccos x, \sqrt{1-x^2} - y)$

$\psi :]-1,1[\times \mathbb{R} \to]0,\pi[\times \mathbb{R}$

$$\forall 0<t<\pi, \quad \psi(\cos t, \sin t) = (t, 0)$$

2°) $\begin{array}{l} \mathbb{R}^2 \xrightarrow{f} \mathbb{R} \\ (x,y) \longmapsto x^2+y^2-1 \end{array}$

Soit $\Phi(x,y) = (\sqrt{x+1-y^2}, y)$

$\Phi : \mathbb{R}_+^* \times]-1,1[\longrightarrow \Phi(\mathbb{R}_+^* \times]-1,1[)$

$$\forall x>0, \forall |y|<1, \quad f(\Phi(x,y)) = x.$$

démo a) $f : \begin{array}{l} \underset{\mathbb{R}^m}{U \text{ ouvert}} \\ U \longrightarrow \mathbb{R}^n \\ a \longmapsto f(a) \end{array}$ $\quad df|_a : \mathbb{R}^m \to \mathbb{R}^n$ injective

$\Rightarrow \operatorname{Jac} f(a) = \left(\partial_{x_j} f_i(a)\right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$ de rang m

On peut supposer $\left(\partial_{x_j} f_i(a)\right)_{1 \leq i,j \leq m}$ inversible

On pose $g(x_1,\dots,x_m,x_{m+1},\dots,x_n) = (f_1(x_1,\dots,x_m),\dots,f_m(x_1,\dots,x_m), f_{m+1}(x) - x_{m+1},\dots, f_n(x) - x_n)$

g est $\mathcal{C}^k$ et $\operatorname{Jac}(g)_{(a,0)} = \left(\begin{array}{c|c} \left(\partial_{x_j} f_i(a)\right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}} & 0 \\ \hline * & -I_{n-m} \end{array}\right)$ inversible

$\Longrightarrow$ (théorème d'inversion locale) $\exists\, (a,0) \in V$ ouvert tq

$$g : \underset{\mathbb{R}^n}{\overset{\cup}{V}} \longrightarrow g(V) \quad C^k \text{ difféomorphisme}$$

et $\forall (x_1, \dots, x_m) \in f^{-1}(V)$, $g^{-1}(f(x_1, \dots, x_m)) = (x_1, \dots, x_m, 0, \dots, 0)$

(car $g(x_1, \dots, x_m, 0, \dots, 0) = (f_1(x), \dots, f_m(x), f_{m+1}(x), \dots, f_n(x)) = f(x)$)

b) Soit $f : \underset{\mathbb{R}^m}{\underset{\text{ouvert}}{U}} \longrightarrow \mathbb{R}^n$ C^k submersion en $a \in U$

$df|_a : \mathbb{R}^m \to \mathbb{R}^n$ surjective $\Leftrightarrow$ $Jac(f)(a) = \left(\partial_{x_j} f_i(a)\right)_{\substack{1 \le i \le n \\ 1 \le j \le m}}$ de rang n

Supposons $\left(\partial_{x_j} f_i(a)\right)_{1 \le i,j \le n}$ inversible

on pose $h : U \longrightarrow \mathbb{R}^m$
$x \longmapsto (f_1(x), \dots, f_n(x), x_{n+1}, \dots, x_m)$

$$Jac(h)(a) = \left(\begin{array}{c|c} (\partial_{x_j} f_i)_{1 \le i,j \le n} & * \\ \hline 0 & I_{n-m} \end{array}\right) \text{ inversible}$$

$\Longrightarrow$ (Inversion locale) $\exists\, a \in V$ ouvert, $h : V \to h(V)$ C^k difféo.

$$f h^{-1}(x_1, \dots, x_m) = (x_1, \dots, x_n)$$

en effet, $h^{-1}(x_1, \dots, x_m) = (t_1, \dots, t_m) \Leftrightarrow (x_1, \dots, x_m) = h(t_1, \dots, t_m)$

$\Leftrightarrow (x_1, \dots, x_m) = (f_1(t), \dots, f_n(t), t_{n+1}, \dots, t_m)$

$\Leftrightarrow x_1 = f_1(t), \dots, x_n = f_n(t)$
$x_{n+1} = t_{n+1}, \dots, x_m = t_m$

$\Rightarrow f h^{-1}(x_1, \dots, x_m) = (f_1(t), \dots, f_n(t)) = (x_1, \dots, x_n).$ $\square$

Généralisation: Théorème Soit $f : \underset{\mathbb{R}^m}{\overset{\text{ouvert}}{U}} \longrightarrow \mathbb{R}^n$ $\mathcal{C}^k$ tq

$$\forall a \in U, \ \text{rang } Jac f(a) = r$$

ALORS il existe $f(a) \in V \subset \mathbb{R}^n$, $0 \in W \subset \mathbb{R}^m$,

$\psi : V \to \psi(V)$, $f(a) \mapsto 0$, $\varphi : W \to \varphi(W) \subset U$, $0 \mapsto a$

$\mathcal{C}^k$ difféo tq $\forall x \in V$, $\psi f \varphi(x) = (x_1, \dots, x_r, 0 \dots 0)$

Pause 5'