Analyse fonctionnelle

Définitions

Soit $f \in \mathcal{L}^1([0, 2\pi])$

on pose $\forall k \in \mathbb{Z}$, $c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} \, dt$

$\forall n \in \mathbb{N}$ $\quad S_n(f)(x) = \sum_{k=-n}^{n} c_k(f) e^{ikx} \quad (\forall x \in \mathbb{R})$

$\forall n \in \mathbb{N}$ $\quad T_n(f) = \frac{S_0(f) + \dots + S_n(f)}{n+1}$

Théorème de Féjer

Si $f \in C^0([0, 2\pi], \mathbb{C})$ et $f(0) = f(2\pi)$

alors $T_n(f) \xrightarrow[n\infty]{} f$ uniformément sur $[0, 2\pi]$

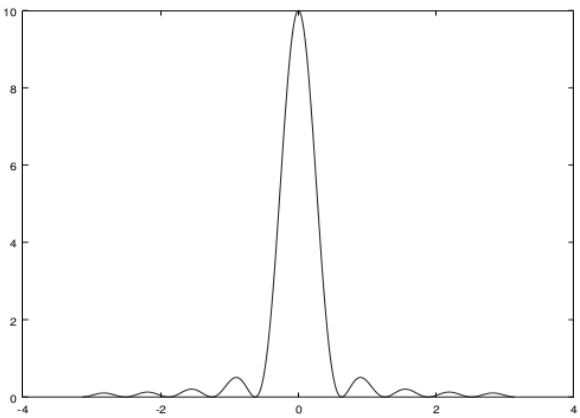
c-à-d : $\|T_n(f) - f\|_\infty \xrightarrow[n\infty]{} 0$

démo:

On pose $\forall x \in \mathbb{R}$, $D_n(x) = \sum_{k=-n}^{n} e^{ikx}$

$F_n(x) = \frac{D_0(x) + \dots + D_n(x)}{n+1}$

Propriétés:



graphe de F_{10}

1) $\forall x \in \mathbb{R}$, $F_n(x) \geq 0$

2) $\forall x \in \mathbb{R}$, $F_n(x) = \begin{cases} \frac{1}{n+1} \frac{\sin^2\left(\frac{n+1}{2} x\right)}{\sin^2\left(\frac{x}{2}\right)} \\ n+1 & \text{si } x \in 2\pi\mathbb{Z} \end{cases}$

3) $\int_0^\pi F_n = \int_{-\pi}^0 F_n = \pi$

4) $\forall\, 0 < \delta < \pi$, $\lim_{n\infty} \sup_{[-\pi,-\delta] \cup [\delta, \pi]} |F_n(x)| = 0$

démo des propriétés

$$\forall k \in \mathbb{N}, \quad D_k(x) = \sum_{l=-k}^{k} e^{ilx} = e^{-ikx} \frac{e^{i(2k+1)x} - 1}{e^{ix} - 1}$$

$$= \frac{\sin\left(k+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}$$

$$\sum_{k=0}^{n} D_k(x) = \frac{1}{\sin\left(\frac{x}{2}\right)} \operatorname{Im}\left(\sum_{k=0}^{n} e^{i\left(k+\frac{1}{2}\right)x}\right)$$

$$= \frac{1}{\sin\left(\frac{x}{2}\right)} \operatorname{Im}\left(e^{ix/2} \left(\frac{e^{i(n+1)x} - 1}{e^{ix} - 1} \right)\right)$$

$$= \frac{1}{\sin\left(\frac{x}{2}\right)} \operatorname{Im}\left(\frac{\cancel{e^{\frac{ix}{2}}}\, e^{i\frac{(n+1)}{2}x}}{\cancel{e^{ix/2}}} \frac{\sin\left(\frac{n+1}{2}x\right)}{\sin(x/2)} \right) = \frac{\sin^2\left(\frac{n+1}{2}\right)x}{\sin^2(x/2)}$$

$$\text{et} \quad \int_0^{\pi} F_n = \frac{1}{n+1} \sum_{k=0}^{n} \underbrace{\int_0^{\pi} D_k}_{\pi} = \pi = \int_{-\pi}^{0} F_n .$$

démo du théorème de Féjer

$$T_n(f)(x) = \frac{1}{n+1} \sum_{k=0}^{n} \sum_{l=-k}^{k} c_l(f) e^{ilx}$$

$$= \frac{1}{n+1} \sum_{k=0}^{n} \sum_{l=-k}^{k} \frac{1}{2\pi} \underbrace{\int_{-\pi}^{\pi} f(t) e^{il(x-t)} dt}_{= \int_{-\pi}^{\pi} f(x-t) e^{ilt} dt}$$

$$\text{Or} \quad \frac{1}{n+1} \sum_{k=0}^{n} \sum_{l=-k}^{k} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ilt} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n = 1$$

$$\text{donc } \forall\, 0 \leq x \leq 2\pi, \quad |T_n(f)(x) - f(x)| = \frac{1}{n+1} \left| \sum_{k=0}^{n} \sum_{l=-k}^{k} \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-t) - f(x)) e^{ilt} dt \right|$$

$$= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-t) - f(x)) F_n(t)\, dt \right|$$

$$\leq \underbrace{\frac{1}{2\pi} \int_{\substack{-\pi \\ |t|<\delta}}^{\pi} |f(x-t) - f(x)| F_n(t) dt}_{} + \frac{1}{2\pi} \int_{\substack{-\pi \\ |t|\geq\delta}}^{\pi} |f(x-t) - f(x)| F_n(t)\, dt$$

Or f est uniformément continue sur $\mathbb{R}$
(car $[0,2\pi]$ compact)

Soit $\varepsilon > 0$, $\exists\, \delta > 0$, $\forall\, |t| < \delta$ $\forall x \in \mathbb{R}$, $|f(x-t) - f(x)| \leq \varepsilon$

$$\Rightarrow \forall x \quad |T_n(f)(x) - f(x)| \leq \frac{1}{2\pi} \varepsilon \int_{-\pi}^{\pi} F_n + \frac{2\|f\|_\infty}{2\pi} \int_{\substack{-\pi \\ |t|\geq\delta}}^{\pi} F_n$$

$$\leq \varepsilon + 2\|f\|_\infty \underbrace{\sup_{\substack{|t|\geq\delta \\ -\pi\leq t\leq\pi}} F_n}_{\xrightarrow[n\infty]{} 0}$$

donc $\limsup_n \|T_n(f) - f\|_\infty \leq \varepsilon \qquad (\forall \varepsilon > 0)$

donc $T_n(f) \longrightarrow f$ uniformément.

Exercice : Soit $f(x) = |\sin x|$

a) Développer en série de Fourier sur $[-\pi, \pi]$

b) En déduire $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

$$c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin x|\, e^{-ikx}\,dx = \frac{1}{2\pi}\int_0^{\pi} \sin x\, e^{-ikx}\,dx + \frac{1}{2\pi}\int_{-\pi}^{0} (-\sin x)\, e^{-ikx}\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin x\left(e^{-ikx} + e^{ikx}\right)dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x \cos kx\,dx$$

or $2\sin x \cos kx = \sin(k+1)x - \sin(k-1)x$

$$c_k(f) = \frac{1}{2\pi}\int_0^{\pi} \sin(k+1)x\,dx - \frac{1}{2\pi}\int_0^{\pi}\sin(k-1)x\,dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(k+1)x}{k+1}\right]_0^{\pi} + \frac{1}{2\pi}\left[\frac{\cos(k-1)x}{k-1}\right]_0^{\pi} \qquad k \neq \pm 1$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{k+1}}{k+1} + \frac{1}{k+1}\right] + \frac{1}{2\pi}\left[\frac{(-1)^{k-1} - 1}{k-1}\right]$$

$$\underset{k=2p}{=} \frac{1}{2\pi}\,\frac{2}{k+1} + \frac{1}{2\pi}\left(\frac{-2}{k-1}\right) = \frac{1}{\pi}\left(\frac{-2}{k^2-1}\right) = \frac{-2}{\pi}\left(\frac{1}{4p^2-1}\right)$$

donc $S_{2n} f(x) = \sum_{p=-n}^{n} \frac{-2}{\pi}\left(\frac{1}{4p^2-1}\right) e^{2ipx}$

$$\xrightarrow[n\infty]{} \frac{f(x^+) + f(x^-)}{2} = f(x)$$ (car f dérivable par morceaux)

donc $x = 0$ $\quad \sum_{p=-\infty}^{\infty} -\frac{2}{\pi}\left(\frac{1}{4p^2-1}\right) = 0 \Rightarrow \frac{2}{\pi} = 2\sum_{p=1}^{\infty}\frac{2}{\pi}\left[\frac{1}{4p^2-1}\right]$

$$\Rightarrow \sum_{p=1}^{\infty} \frac{1}{4p^2-1} = \frac{1}{2}$$

$\square$

Dernier cours jeudi 7 mai à 14H.