Probabilités

Loi Gaussienne

X v.a. réelle suit une loi gaussienne $N(m,\sigma^2)$ si

$$E(f(X)) = \frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} f(x) e^{-\frac{(x-m)^2}{2\sigma^2}} dx \qquad (\forall f \in C_b(\mathbb{R}))$$

<u>Propriétés</u> $E(X) = m$

$\sigma^2(X) = \sigma^2$

Définition. Soit X v.a. réelle, on note $\Phi_X(\xi) = E(e^{i\xi . X})$ (fonction caractéristique)

<u>ex</u> si $X \sim N(0,\sigma^2)$ alors $\Phi_X(\xi) = e^{-\frac{\sigma^2\xi^2}{2}}$

démo : $$\frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} e^{ix\xi} e^{-\frac{x^2}{2\sigma^2}} dx = \Phi(\xi)$$

$$\Phi'(\xi) = \frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} ix e^{ix\xi} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{-1}{\sqrt{2\pi}\sigma}\cdot\sigma^2 \int_{\mathbb{R}} \xi e^{-\frac{x^2}{2\sigma^2}} e^{ix\xi} dx$$

$u' = ixe^{-\frac{x^2}{2\sigma^2}}$ $\quad v = e^{ix\xi}$ $\qquad = -\xi\sigma^2\Phi(\xi)$

$u = -\sigma^2 i e^{-\frac{x^2}{2\sigma^2}}$ $\quad v' = i\xi e^{ix\xi}$

$$\Phi'(\xi) = \sigma^2\xi\Phi(\xi) \Rightarrow \Phi(\xi) = \Phi(0) e^{-\frac{\sigma^2\xi^2}{2}} = e^{-\frac{\sigma^2\xi^2}{2}}$$

Proposition : « Φ_X caractérise X »

si X_1 v.a de loi avec densité f

si X_2 v.a " " avec densité g

$$\Phi_{X_1} = \Phi_{X_2} \Rightarrow \forall\varphi \in C_b^0(\mathbb{R}), \int_{\mathbb{R}} f(x)\varphi(x)dx = \int_{\mathbb{R}} g(x)\varphi(x) dx$$

$$E(\varphi(X_1)) = E(\varphi(X_2)).$$

<u>démo</u>. $$p_\sigma(x) := \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

$$f_\sigma = p_\sigma * f \qquad f_\sigma(x) = \int_{\mathbb{R}} p_\sigma(x-y) f(y)\, dy$$

1) $\forall\varphi \in C_b(\mathbb{R})$, $$\lim_{\substack{\sigma\to 0\\ \sigma>0}} \int_{\mathbb{R}} f_\sigma(x)\varphi(x) dx = \int_{\mathbb{R}} f(x)\varphi(x) dx$$

En effet, $$\int_{\mathbb{R}} f_\sigma(x)\varphi(x)dx = \int_{\mathbb{R}}\int_{\mathbb{R}} p_\sigma(x-y) f(y)\varphi(x)\, dx\, dy = \frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2\sigma^2}}\varphi(x) f(y)\, dx\, dy$$

$$\underset{t=\frac{x-y}{\sigma}}{=} \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\Big(\underbrace{\int_{\mathbb{R}} e^{-\frac{t^2}{2}}\varphi(\sigma t+y)\, dt}_{\xrightarrow[\sigma\to 0]{}\varphi(y)}\Big) f(y)\, dy$$

$$\xrightarrow[\sigma\to 0]{} \int_{\mathbb{R}} f(y)\varphi(y) dy \quad \text{(convergence dominée)}$$

De même : $\lim_{\sigma\to 0}\int_{\mathbb{R}} g_\sigma(x)\varphi(x)dx = \int_{\mathbb{R}} g(x)\varphi(x)dx$

si $g_\sigma = p_\sigma * g$.

2) Posons $\widehat{p_\sigma}(\xi) = e^{-\frac{\sigma^2\xi^2}{2}} = \sqrt{2\pi}\, p_{1/\sigma}(\xi)$

On a vu que $\widehat{p_\sigma}(\xi) = \int_{\mathbb{R}} e^{ix\xi} p_\sigma(x)\, dx$

En particulier, $p_\sigma(\xi) = \frac{1}{\sqrt{2\pi}} \widehat{p_{1/\sigma}}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{it\xi} p_{1/\sigma}(t)\, dt$

Soit $\varphi \in C_b^0(\mathbb{R})$. $\int_{\mathbb{R}} f_\sigma(x)\varphi(x)\, dx = \int_{\mathbb{R}}\int_{\mathbb{R}} p_\sigma(x-y) f(y)\varphi(x)\, dy\, dx = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{it(x-y)} p_{1/\sigma}(t) f(y)\varphi(x)\, dx\, dy\, dt$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{itx} p_{1/\sigma}(t)\varphi(x) \underbrace{\left(\int_{\mathbb{R}} e^{-ity} f(y)\, dy\right)}_{\Phi_{X_1}(-t)} dx\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{itx} p_{1/\sigma}(t)\Phi_{X_1}(-t)\, dt\right) \varphi(x)\, dx$$

Comme $\Phi_{X_1} = \Phi_{X_2}$ on a $\int_{\mathbb{R}} f_\sigma(x)\varphi(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{itx} p_{1/\sigma}(t)\, \Phi_{X_2}(-t)\, dt\right)\varphi(x)\, dx$

$$= \int_{\mathbb{R}} g_\sigma(x)\varphi(x)\, dx$$

si on pose $g_\sigma = p_\sigma * g$.

Donc $\forall \varphi \in C_b^0(\mathbb{R})$, $\int_{\mathbb{R}} f(x)\varphi(x)\, dx = \lim_{\substack{\sigma \to 0 \\ \sigma > 0}} \int_{\mathbb{R}} f_\sigma(x)\varphi(x)\, dx$

$$= \lim_{\substack{\sigma \to 0 \\ \sigma > 0}} \int_{\mathbb{R}} g_\sigma(x)\varphi(x)\, dx$$

$$= \int_{\mathbb{R}} g(x)\varphi(x)\, dx \qquad \square$$