IV. 4 Plus d'Exemples

18/04
Gricey
ANALYSE 2

ou → **Sol²Ex** Pendant le CM 11
Fiche de ~~TD~~, Ex. du CM10 (le dernier)

Exple: ① $y'' - 2y' + 5y = 5 \cdot e^{2x} \cdot \cos x$, $y(0) = 3$, $y'(0) = \frac{5}{2}$

1) la dernier fois : $y_{hom}(x) = A \cdot e^{x} \cdot \cos(2x) + B \cdot e^{x} \cdot \sin(2x) \equiv e^{x}\left(A \cdot \cos(2x) + B \sin(2x)\right)$, $A, B \in \mathbb{R}$

2) —— " —— : $y_{part}(x) = e^{2x} \cdot \cos x + \frac{1}{2} e^{2x} \cdot \sin(2x) \equiv e^{2x}\left(\cos x + \frac{1}{2} \sin x\right)$

3) la solution générale: $y(x) \equiv y_{gen}(x) = y_{hom}(x) + y_{part}(x) = e^{x}\left(A \cdot \cos(2x) + B \sin(2x)\right) + e^{2x}\left(\cos x + \frac{1}{2}\sin x\right)$

$$\underbrace{e^{(0)}}_{1}\left(A \cdot \underbrace{\cos(0)}_{1} + B \underbrace{\sin(0)}_{0}\right) + \underbrace{e^{(0)}}_{1}\left(\underbrace{\cos(0)}_{1} + \frac{1}{2}\underbrace{\sin(0)}_{0}\right)$$

4) $y(0) = A + 1 \overset{!}{=} 3 \iff \boxed{A = 2}$

car $\exp(0) = 1$, $\cos(0) = 1$, $\sin(0) = 0$.

$y'(0) \overset{*}{=} A + 2B + 2 + \frac{1}{2} \overset{!}{=} \frac{5}{2} \iff \boxed{B = -1}$

On observe. Je ne calcule pas $y'(x)$ mais directement $y'(0)$!

(*) $f(x) = e^{\alpha \cdot x} \cdot \cos(\beta x)$

$f'(x) = \alpha \cdot e^{\alpha x} \cdot \cos(\beta x) - e^{\alpha x} \cdot \beta \sin(\beta x)$

$\boxed{f'(0) =} \quad \alpha \cdot 1 \quad 1 \quad + \quad 0 \quad \boxed{= \alpha}$

$g(x) = e^{\alpha x} \cdot \sin(\beta x)$

$g'(x) = \alpha \cdot e^{\alpha x} \cdot \underbrace{\sin(\beta x)}_{0} + \beta \cdot \underbrace{e^{\alpha x} \cdot \cos(\beta x)}_{1}$, $\boxed{g(0) = \beta}$

Alors, la solution (satisfaisant les conditions initiales):

$$y(x) = e^{x}\left[2 \cdot \cos(2x) - \sin(2x)\right] + e^{2x}\left(\cos x + \frac{1}{2}\sin x\right)$$

IV. 4.1 d'ordre 1

Exple: ① $\boxed{\cos x \cdot y' - \sin x \cdot y = \cos x}$, $\boxed{y\left(\frac{3\pi}{4}\right) = 1}$ (* *)

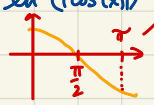
(*) bien défini $\forall x \in \mathbb{R}$.

≡ ÉTAPES ≡

$\frac{3\pi}{4} \in \left]\frac{\pi}{2}, \pi\right] =: I$ et $\cos\big|_I$ divise, alors $\forall x \in I$: (*) $\iff$ $\boxed{y' - \tan x \cdot y = 1}$ (*')

① Pb. Hom ⟶ $a(x) = -\tan(x) = -\frac{\sin(x)}{\cos(x)} \Longleftarrow A(x) = \ln(|\cos(x)|) = \ln(-\cos(x))$ $\forall x \in I$

$\frac{u'}{u}$ et $\cos'(x) = -\sin(x)$



$y_{hom}(x) = C \cdot e^{-A(x)} = C \cdot \exp(-\ln(-\cos x)) = C \cdot e^{\ln[(-\cos x)^{-1}]} = \frac{-C}{\cos x} =: \frac{\tilde{C}}{\cos x}$

② Sol. part. Variation de la constante.

$y(x) := C(x) \cdot \frac{1}{\cos x}$

$\Rightarrow y' - \tan(x) \cdot y = \frac{C'(x)}{\cos x}$

$\Rightarrow C'(x) = \cos(x)$

$C(x) = \sin(x) \Rightarrow \boxed{y_{part}(x) = \frac{\sin x}{\cos x} = \tan x}$

Rappel: variation de la constante :

$y_h(x)$ une solution du pb. hom.,

$y(x) = C(x) \cdot y_h(x)$

$\alpha y' + \beta y \equiv L[y] \overset{!}{=} b \Longrightarrow \boxed{\alpha(x) \cdot C'(x) \cdot y_h(x) = b(x)}$

$L = \alpha(x) \frac{d}{dx} + \beta(x)$ (ici $\alpha = \cos x$, $\beta = -\sin x$, $b = \cos x$ ou $\alpha = 1$, $\beta = \tan x$, $b = 1$)

③ $y(x) = y_{gen}(x) = y_{hom}(x) + y_{part}(x) =$

$= \dfrac{\tilde{C}}{\cos x} + \tan(x), \quad \tilde{C} \in \mathbb{R}$

La cond. init: $y\left(\frac{3\pi}{4}\right) = \dfrac{\tilde{C}}{\cos\left(\frac{3\pi}{4}\right)} + \underbrace{\tan\left(\frac{3\pi}{4}\right)}_{\substack{\tan\left(\frac{3\pi}{4}-\pi\right) \\ = -\tan\frac{\pi}{4} = 1}} = \dfrac{\tilde{C}}{-\frac{1}{\sqrt{2}}} - 1 \equiv -\sqrt{2}\,\tilde{C} - 1 \overset{!}{=} 1 \Rightarrow \tilde{C} = -\sqrt{2}$

$\cos\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{3\pi}{4} - \pi\right) = -\cos\left(\pm\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$$\boxed{y(x) = \frac{-\sqrt{2}}{\cos x} + \tan x}$$

Exple: ② $y' - 2\tan(x) \cdot y = 2, \quad y\left(\frac{\pi}{4}\right) = 1 \rightarrow x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

① $a(x) = -2\tan(x)$
$A = 2 \cdot \ln(|\cos(x)|) = 2\ln(\cos x)$

$$\boxed{y_{hom}(x) = C \cdot e^{-2\ln(\cos x)} = \frac{C}{\cos^2 x}}$$

② $y = \dfrac{C(x)}{\cos^2(x)} \Rightarrow \dfrac{C'(x)}{\cos^2 x} = 2$

$C'(x) = 2 \cdot \cos^2(x)$

$$\boxed{y_{part}(x) = \frac{\cancel{\frac{1}{2}}\sin x \cdot \cos x + \cancel{\frac{1}{2}}x}{\cos^2 x} = \cancel{\frac{1}{2}}\tan x + \cancel{\frac{1}{2}}\frac{x}{\cos^2 x}}$$

(IPP ou CdV !)

$\displaystyle\int \cos^2 x \, dx \overset{IPP}{=} \cos x \cdot \sin x + \int \underbrace{\sin^2 x}_{1-\cos^2 x} \, dx \Longrightarrow 2\int \cos^2 x \, dx = \cancel{\tfrac{1}{2}}(\cos x \cdot \sin x + x)$

$u = \cos x \quad v' = \cos x$

plus simple : utiliser $2\cos^2 x = \cos(2x) + 1$

③ $y(x) = \tan(x) + \dfrac{x + C}{\cos^2 x}, \quad C \in \mathbb{R}$

$y\left(\frac{\pi}{4}\right) = 1 + \dfrac{\frac{\pi}{4} + C}{\frac{1}{2}} \overset{!}{=} 1 \Longleftrightarrow C = -\frac{\pi}{4}$

$$\boxed{y(x) = \tan x + \frac{x - \frac{\pi}{4}}{\cos^2 x}}$$

Verification: $\tan\frac{\pi}{4} = 1$, $\cos^2 x$: 1, 0

Exple: ③ (d'ordre 2)

$y'' + 8y' + 25y = x \cdot e^x \qquad y(0) = \qquad , y'(0) =$

① pb. Hom

$y' + 8y' + 25y = 0 \quad , \quad y(x) = e^{\lambda \cdot x}$

$\lambda^2 + 8\lambda + 25 = 0$

$y' = \lambda \cdot y$

$y'' = \lambda^2 \cdot y$

$\lambda_{1,2} = -4 \pm \sqrt{16 - 25}$

$\boxed{\lambda_{1,2} = -4 \pm 3i}$

Re, Im, Re, Im

$y_{hom}(x) = A \cdot e^{-4x} \cdot \cos(3x) + B e^{-4x} \cdot \sin(3x)$

Rappel: $e^{(-4+3i)x} = e^{-4x}\left[\cos(3x) + i\sin(3x)\right]$

② $y(x) = (\alpha \cdot x + \beta)e^x, \quad y' = (\alpha \cdot x + \beta + \alpha)e^x$

$y''(x) = (\alpha \cdot x + \beta + 2\alpha)e^x$

$y'' + 8y' + 25y = e^x\left[\underbrace{(\alpha x + \beta + 2\alpha) + 8(\alpha x + \beta + \alpha) + 25(\alpha x + \beta)}_{34\alpha x + 34\beta + \cancel{35\alpha}\ 10\alpha}\right] \overset{!}{=}$

$\overset{!}{=} e^x \cdot x$

$\Longleftrightarrow 34\alpha \overset{!}{=} 1, \quad 34\beta + \cancel{35}\,10\,\alpha \overset{!}{=} 0 \Longleftrightarrow \boxed{\begin{array}{l} \alpha = \frac{1}{34} \\ \beta = -\frac{\cancel{35}\,10}{(34)^2} \end{array}}$

$$\boxed{y_{part}(x) = \frac{1}{34}\left(x - \frac{\cancel{35}\,10}{34}\right)e^x}$$

3 $y_{gen} = y_{hom} + y_{part}$

$$y(x) = \underline{e^{-4x}\left(A \cdot \cos(3x) + B \sin(3x)\right)} + \frac{1}{34}\left(x - \frac{5}{17}\right)e^{x}$$

$$y(0) = A - \frac{10}{(34)^2} \overset{!}{=} 0 \iff A = \frac{10}{(34)^2} \equiv \frac{5}{2 \cdot (17)^2}$$

$$y'(0) = -4A + 3B + \frac{1}{34}\underbrace{\left(1 - \frac{5}{17}\right)}_{\frac{12}{17}} \overset{!}{=} 0$$

$$B = \frac{1}{3}\left(4A - \frac{6}{(17)^2}\right) = \frac{1}{3} \quad \frac{10 - 6}{(17)^2} \Longrightarrow B = \frac{4}{3(17)^2}$$