14/03 Gracey ANALYSE 2

Ex 2. $\int \sin(2x)e^{3x}\,dx =$

$\rightsquigarrow ((A\sin(2x)+B\cos(2x))e^{3x})' = ((3A-2B)\sin(2x)+(3B+2A)\cos(2x))e^{3x} = \sin(2x)e^{3x}$

$\Leftrightarrow \begin{cases} 3A-2B \\ 2A+3B \end{cases} \Leftrightarrow \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} A \\ B \end{pmatrix} = 1/3 \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1/13\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ $B = \frac{2}{5}$ $A = \frac{-3}{5}$

Rq: Pour $\sin(\beta x)e^{\alpha x}$ et/ou $\cos(\beta x)e^{\alpha x}$, on cherche une solution de la forme : $A\sin(\beta x)e^{\alpha x} + B\cos(\beta x)e^{\alpha x}$

III.4 Intégration de fractions rationelles (continuation)

Soit $\int \frac{P(x)}{Q(x)}dx$ **Étape 1:** Division Euclidienne et DES du reste.

$$\underbrace{\int \tilde{P}(x)\,dx}_{1} + \underbrace{\int \frac{A}{x-c_1}dx + \int \frac{\tilde{A}}{(x-c_1)}dx}_{2} + \dots + \underbrace{\int \frac{B+Cx}{x^2+px+q}dx + \int \frac{\tilde{B}+\tilde{C}x}{(x^2+px+q)^2}dx}_{3}$$

$\hookrightarrow \Delta = p^2-4q<0$

Pour 1 et 2 $\int x^\alpha dx = \begin{cases} \ln(|x|)+c, & \text{si } \alpha=-1 \\ \frac{x^{\alpha+1}}{\alpha+1}+c, & \text{si } \alpha\neq -1 \end{cases} \Rightarrow$ 1. $\int (a_0+a_1\cdot x+a_2\cdot x^2+\dots+a_n\cdot x^n)\cdot dx = \sum_{k=0}^{n} a_k \int x^k dx = \sum_{k=0}^{n} a_k x^k$

$n\geq 1,$ $n\geq 2$ $\int x^{1/n}dx = \int x^{-n}dx = \frac{x^{-n+1}}{-n+1}+C = \frac{1}{1-n}\cdot\frac{1}{x^{n-1}}+c \equiv -\frac{1}{n-1}\cdot\frac{1}{x^{n-1}}+c$

$\Rightarrow$ 2 i) $\int \frac{1}{x-c}dx = \ln(|x-c|)+C$ $\hookleftarrow$ avec CdV, $u=x-c$, $du=dx$

ii) $\int \frac{1}{(x-c)^n} = \frac{-1}{(x-c)^{n-1}}+C$, pour $n\geq 2$

Rq: $\frac{1}{x^n}dx = \frac{-1}{n-1}\cdot\frac{1}{x^{n-1}}+c$

$n=2$: $\int \frac{1}{x^2}\cdot dx = -\frac{1}{x}+c$

$n=3$: $\int \frac{1}{x^3}\cdot dx = -\frac{1}{2}\cdot\frac{1}{x^2}+c$

Pour 3 Stratégie i) $\frac{1}{x^2+px+q}dx \xrightarrow{\text{cdv}} \frac{1}{1+x^2}dx$

ii) $\frac{x}{x^2+px+q}dx = \frac{1}{2}\underbrace{\int \frac{2x+p}{(x^2+px+q)^n}dx}_{*} - \frac{p}{2}\underbrace{\int \frac{1}{(x^2+px+q)}dx}_{*}$

Rq: $(x^2+px+q)' = 2x+p$

$*\int \frac{2x+p}{(x^2+px+q)^n}dx = \int \frac{\varphi'(x)}{\varphi(x)^n}dx = \frac{-1}{n-1}\cdot\frac{1}{\varphi(x)^{n-1}}+C \equiv \frac{-1}{n-1}\cdot\frac{1}{(x^2+px+q)^{n-1}}+C$

$*\int \frac{1}{(x^2+px+q)^2}dx = \begin{cases} \text{pour } n=1 \longrightarrow i) \\ \text{pour } n>1 \text{ plus tard} \end{cases}$

Ex: $I = \int_0^2 \frac{3}{x^3+1}dx$

ét 1 : décomp. $\frac{3}{x^3+1}$ en élém. simple :

$x^3+1 = (x+1)(x^2-x+1)$, $\Delta=-3$

$\frac{3}{x^3+1} = \frac{A}{x+1} + \frac{B+Cx}{x^2-x+1} = \frac{1}{x+1} + \frac{2-x}{x^2-x+1}$

$A = \left.\frac{3}{x^2-x+1}\right]_{x=-1} = 1$ $\quad \frac{3}{2} = \frac{1}{2}+2+\frac{C}{2}$ $x=1$ $\Rightarrow C=-1$

$A+B=3$ $x=0$

$B=2$

$I = \ln(|x+1|)\Big]_0^2 - \frac{1}{2}\int_0^2 \frac{2x}{x^2-x+1}dx + 2\int_0^2 \frac{1}{x^2-x+1}dx$

$= \ln(3) - \frac{1}{2}\int_0^2 \frac{2x+1}{x^2-x+1}dx + \frac{3}{2}\underbrace{\int_0^2 \frac{1}{x^2-x+1}dx}_{=J} = \ln(\sqrt{3})+\pi$

$\int_0^2 \frac{2x-1}{x^2-x+1}dx = \ln(x^2-x+1)+C$, $u=x^2-x+1$ pour $x\in[0,2]$.

$I = \ln(3) - \frac{1}{2}\left[\ln(x^2-x+1)\right] + \frac{3}{2}J$

$= \ln(3) - \frac{1}{2}\ln(3) + \frac{3}{2}J = \frac{1}{2}\ln(3) + \frac{3}{2}J = \ln(\sqrt{3}) + \frac{3}{2}J = \ln(\sqrt{3})+\pi$

où $J := \int_0^2 \frac{1}{x^2-x+1}dx = \int_{-1/2}^{3/2} \frac{1}{u^2+\frac{3}{4}}du = \frac{4}{3}\int_{-1/\sqrt{3}}^{\sqrt{3}} \frac{1}{v^2+1}dv = \frac{4}{3}\Big[\arctan(v)\Big]_{-1/\sqrt{3}}^{\sqrt{3}}$

$= 4/3\Big(\underbrace{\arctan(\sqrt{3})}_{=\pi/3} + \underbrace{\arctan(1/\sqrt{3})}_{=\pi/3}\Big) = \frac{2\pi}{3}$ $\longrightarrow -\arctan(-1/\sqrt{3}) = \arctan(1/\sqrt{3})$

① $u = x-1/2$ $\hookrightarrow$ bijectᵒ ② $du = du$ ③

x	2	0
u	$3/2$	$-1/2$

$u^2+3/4 = 3/4\,(\underbrace{4/3\,u^2}_{=v^2}+1)$

① $v = \frac{2}{\sqrt{3}}u$ ② $du = dv$

u	$3/2$	$\sqrt{3}$
v	$-1/2$	$-1/\sqrt{3}$

Rappel: $\displaystyle\int_a^b \frac{1}{(x-c)^n}\cdot dx = \begin{cases}[\ln|x-c|]_a^b, & \text{si } n=1\\ \left[\frac{-1}{n-1}\cdot\frac{1}{(x-c)^{n-1}}\right]_a^b, & \text{si } n>1\end{cases}$ (avec $c \notin [a,b]$)

$I_n(A,B) := \displaystyle\int \frac{A+Bx}{(x^2+px+q)^n}\cdot dx$ t.q. $\Delta \equiv p^2-4q<0$

Toutes les intégrales des éléments simples.

Cas 1: $B\neq 0$ (but → réduire au cas $B=0$)

$$\boxed{I_n(A,B) = \int_a^b \frac{Bx+A}{(x^2+px+q)^n}dx} = \boxed{A I_n(1,0) + B I_n(0,1)}$$

$(x^2+px+q)' = 2x+p$

CdV: $u = x^2+px+q$, $du=(2x+p)dx$

si $n>1$

$$I_n(0,1) = \frac12\int_a^b \frac{2x+p}{(x^2+px+q)^n}dx - \frac12 I_n(p,0) = \underbrace{\frac12 \int_{a^2+pa+q}^{b^2+pb+q}\frac{1}{u^n}du}_{\frac12\left[-\frac{1}{n-1}\cdot\frac{1}{u^{n-1}}\right]_{u=a^2+pa+q}^{b^2+pb+q}} - \frac12 I_n(p,0)$$

$$I_n(0,1) = \begin{cases} -\frac{1}{2(n-1)}\left[\frac{1}{(x^2+px+q)^{n-1}}\right]_{x=a}^b - \frac p2 I_n(1,0)\\ \frac12\left[\ln(\underbrace{x^2+px+q}_{(\Delta<0\Rightarrow)>0})\right]_a^b - \frac p2 I_n(1,0)\end{cases}$$

Alors il reste $I_n(1,0)$:

$$I_n(1,0) \equiv \int_a^b\frac{1}{(x^2+px+q)^n}dx =: J_n(p,q)$$

but : réduire au cas $J_n(0,1)$ par CdVs

$$x^2+px+q = \left(x+\frac p2\right)^2 - \left(\frac p2\right)^2 + q \equiv \left(x+\frac p2\right)^2 - \frac{\Delta}{4} = \left(x+\frac p2\right)^2 + \frac{|\Delta|}{4}$$

car $\Delta \equiv p^2-4q<0$

1er CdV: $u = x + \frac{p}{2}$, $du = dx$

$$\boxed{J_n(p,q)} = \int_{a+p/2}^{b+p/2}\frac{1}{\left(u^2+\frac{|\Delta|}{4}\right)}du \equiv \int_{a+\frac p2}^{b+p/2}\frac{1}{\left(\frac{|\Delta|}{4}-x^2\right)^n}dx = \boxed{\left(\frac{4}{|\Delta|}\right)^n\int_{a+p/2}^{b+p/2}\frac{1}{\left(1+\frac{4}{|\Delta|}x^2\right)^n}dx}$$

$\frac{|\Delta|}{4}+x^2 = \frac{|\Delta|}{4}\left(1+\frac{4}{|\Delta|}x^2\right)$

2ème CdV: $u = \frac{2x}{\sqrt{|\Delta|}}$, $du = \frac{2}{\sqrt{|\Delta|}}dx$ $\quad \left[\tilde x := \frac{2\hat x}{\sqrt{|\Delta|}} = \frac{2x+p}{\sqrt{|\Delta|}}\right]$

$\hat x = x + \frac p2$

x	u
b	$(2b+p)/\sqrt{\vert\Delta\vert} \equiv \tilde b$
a	$(2a+p)/\sqrt{\vert\Delta\vert} \equiv \tilde a$

$$\boxed{J_n(p,q)} = \left(\frac{4}{|\Delta|}\right)^n\int_{\hat a}^{\hat b}\frac{1}{\left(1+\left(\frac{2x}{\sqrt{|\Delta|}}\right)^2\right)^n}dx = \left(\frac{4}{\sqrt{|\Delta|}}\right)^n\cdot\frac{2}{\sqrt{|\Delta|}}\cdot\int_{\tilde a}^{\tilde b}\frac{1}{(1+u^2)^n}du = \boxed{\frac{2^{2n+1}}{\left(\sqrt{|\Delta|}\right)^{n+1}}\cdot\int_{\frac{(2a+p)}{\sqrt{|\Delta|}}}^{\frac{2b+p}{\sqrt{|\Delta|}}}\frac{1}{(1+x^2)^n}dx}$$

$\equiv$ $I_n(1,0)$

Il reste:

$$K_n := \int_a^b\frac{1}{(1+x^2)^n}dx \quad \left(\equiv K_n(a,b) \longrightarrow I_n(1,0) = \frac{2^{2n+1}}{|\Delta|^{\frac{n+1}{2}}}K(\tilde a,\tilde b) \text{ où } \tilde a = \frac{2a+b}{\sqrt{|\Delta|}}, \tilde b = \frac{2b+p}{\sqrt{|\Delta|}}\right)$$

$$K_1 = [\arctan(x)]_a^b$$

K_n pour $n>1$, par une récursion :

$$K_n \equiv \int_a^b\frac{1\overset{+x^2-x^2}{}}{(1+x^2)^n}dx = K_{n-1} - \frac12\int_a^b x\,\frac{2x}{(1+x^2)^n}dx = K_{n-1} + \frac{1}{2(n-1)}\left[\frac{x}{(1+x^2)^{n-1}}\right]_a^b - \underbrace{\frac12\int\frac{1}{n-1}\cdot\frac{1}{(1+x^2)^{n-1}}dx}_{\frac{1}{2(n-1)}\cdot K_{n-1}}$$

$$K_n = \frac{1}{2(n-1)}\left(-K_{n-1} + \left[\frac{x}{(1+x^2)^{n-1}}\right]_a^b\right)$$

$$\ast\begin{cases}u = x & v' = \frac{2x}{(1+x^2)^n}\\ u'=1 & v = -\frac{1}{n-1}\cdot\frac{1}{(1+x^2)^n}\end{cases}$$