3) $I = \int_0^1 \underbrace{x}_{u'} \underbrace{\arctan(x)}_{v}\, dx = \left[\frac{x^2}{2}\arctan(x)\right]_0^1 - \frac{1}{2}\int_0^1 x^2 \frac{1}{1+x^2}\, dx =$

$u' = x \qquad v = \arctan(x)$

$u = \frac{x^2}{2} \qquad v' = \frac{1}{x^2+1}$

$= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\underbrace{\int_0^1 \frac{x^2+1-1}{x^2+1}\, dx}_{\int_0^1 1\, dx - \int_0^1 \frac{1}{1+x^2}\, dx} = \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right) = \boxed{\frac{\pi}{4} - \frac{1}{2}}$

Ou un peu plus rapide:
un autre choix de primitive pour x

$I = \int_0^1 x \underbrace{\arctan(x)}_{v}\, dx = \left[\frac{x^2+1}{2}\arctan(x)\right]_0^1 - \frac{1}{2}\underbrace{\int_0^1 (x^2+1)\cdot\frac{1}{1+x^2}\, dx}_{1} =$

$u = \frac{x^2+1}{2} \qquad v = \arctan(x)$

$\left[\frac{x^2+1}{2}\arctan(x)\right]_0^1 \to \frac{\pi}{4}$, $\quad -\frac{1}{2}\cdot 1$

$= \boxed{\frac{\pi}{4} - \frac{1}{2}}$

4) $\int \sin^2 x\, dx = \int \underbrace{\sin x}_{u} \cdot \underbrace{\sin x}_{v'}\, dx$

$u = \sin x \qquad v' = \sin x$

$u' = \cos x \qquad v = -\cos x$

$= -\sin x\cdot\cos x + \int \underbrace{\cos^2 x}_{1-\sin^2 x}\, dx = -\sin x \cdot \cos x + x - \int \sin^2 x\, dx$

$\boxed{= \int \sin^2 x\, dx = \frac{1}{2}\left(-\sin x\cdot\cos x + x\right) + C}$

III.3.2 Changement de Variable (CdV)

$[F(\varphi(x))]' = F'(\varphi(x))\cdot\varphi'(x)$ (*) $= f(\varphi(x))\cdot\varphi'(x)$ si $F' = f$

Thm: (CdV): Soit $f \in C^0(I)$, $\varphi \in C^1([a,b])$ t.q. $\varphi([a,b]) \subset I$, alors

1) $\boxed{\int_a^b f(\varphi(x))\cdot\varphi'(x)\, dx = \int_{\varphi(a)}^{\varphi(b)} f(x)\, dx}$

2) et si φ est bijective: $\boxed{\int_A^B f(x)\, dx = \int_{\varphi^{-1}(A)}^{\varphi^{-1}(B)} f(\varphi(x))\cdot\varphi'(x)\, dx}$

Preuve: 1) $\int_a^b f(\varphi(x))\cdot\varphi'(x)\, dx =$

$\overset{(*)}{=} \int_a^b [F(\varphi(x))]'\, dx = [F(\varphi(x))]_{x=a}^{b} = F(\varphi(b)) - F(\varphi(a)) = [F(u)]_{u=\varphi(a)}^{\varphi(b)} = \int_{\varphi(a)}^{\varphi(b)} f(x)\, dx$ $\square$.

en utilisant l'éq. (*)

2) $A = \varphi(a) \iff a = \varphi^{-1}(A)$, $B = \varphi(b) \iff b = \varphi^{-1}(B)$ $\square$.

Exple: $\boxed{\int_\alpha^\beta \cos x\cdot e^{\sin x}\, dx} = \int_{\sin(\alpha)}^{\sin(\beta)} e^x\, dx = [e^x]_{\sin\alpha}^{\sin\beta} = \boxed{e^{\sin\beta} - e^{\sin\alpha}}$

de 1) où φ n'est pas une bijection

$f(x) = \exp(x) \equiv e^x$
$\varphi(x) = \sin(x)$

$e^{\sin x}\cdot\cos x = \exp(\varphi(x))\cdot\varphi'(x)$

Notation: / des étapes par 2):

$$\int_A^B f(u)\,du = \int_{\varphi^{-1}(A)}^{\varphi^{-1}(B)} \underbrace{f(\varphi(x))}_{f(u)}\ \underbrace{\varphi'(x)\,dx}_{du} \equiv \int_A^B f(u)\,du$$

étape

1) $u = \varphi(x)$ nouvelle variable $\Updownarrow$ $x = \varphi^{-1}(u)$

2) $du \overset{\text{Déf.}}{=} \varphi'(x)dx \iff dx = (\varphi^{-1}(u))'\,du = \dfrac{1}{\varphi'(\varphi^{-1}(u))}\,du$

3) Bornes

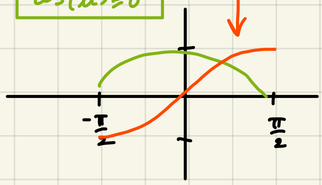
x	u
$\varphi^{-1}(B)$	B
$\varphi^{-1}(A)$	A

CdV 2) **Exple:** $I = \int_{-1}^{1} \underbrace{\sqrt{1-x^2}}_{\cos(u)}\ \underbrace{dx}_{\cos(u)\,du}$ * il faut changer aussi x et bornes !

⚠ $\sqrt{y^2} = \underbrace{|y|}_{\text{module}}$

sin $|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$

ici $\cos(u) \geq 0$



$[-\frac{\pi}{2}, \frac{\pi}{2}] \xrightarrow{\sim} [-1, 1]$ une bijection

étape 1 : $\underbrace{x = \sin(u)}_{\text{Nouvelle variable}} \implies 1 - x^2 = 1 - \sin^2 u = \cos^2(u)$

$f(x) = \sqrt{1-x^2} = \sqrt{\cos^2(u)} = |\cos(u)|$

$= \cos(u)$ pour $u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

étape 2 : $dx = d(\sin(u)) = \sin'(u) \cdot du = \cos(u)\,du$

étape 3:

x	u
1	$\pi/2$
-1	$-\pi/2$

$$I = \int_{-1}^{1} \sqrt{1-x^2}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\cos^2(u)}_{= \frac{1}{2}}\,du = \frac{\pi}{2}$$

$\cos^2 + \sin^2$

Rq: CdV $x \leftrightarrow u$ sans bijection pour trouver des primitives, possible mais avec prudence ⚠

Exple: ci-dessous

Rq: $J := \int_\alpha^{\alpha + k\cdot\frac{\pi}{2}} \cos^2 x\,dx = \int_\alpha^{\alpha+k\frac{\pi}{2}} \underbrace{\sin^2 x}_{1-\cos^2 x}\,dx \implies 2J = \int_0^{\alpha+k\frac{\pi}{2}} 1\,dx = k\,\frac{\pi}{2} \implies J = \frac{k\frac{\pi}{2}}{2}$

$k \in \mathbb{Z}^*$

OU $\int_0^\pi \cos^2 x\,dx = \int_0^\pi \left(\frac{\cos^2 x}{2} + \frac{\cos^2 x}{2}\right) dx = \int_0^\pi \frac{\cos^2 x + \sin^2 x}{2}\,dx = \frac{\pi}{2}$

Exple: (Rq) $F(x) = \int e^{\sin x} \cos(x)\,dx = \int e^u\,du = e^u + C \overset{!}{=} e^{\sin x} + C$

$u = \sin x$

$du = \cos x\,dx$

III.3.3 Chercher primitive de la forme ...

Rq: D'intégrer est un peu un art (pas de procédure fixé) et souvent $\exists$ plusieurs méthodes pour y arriver (dont IPP, CdV).

Exple:

1) $\int x^2 e^{2x}\,dx$ $\quad x^2 e^{2x} \to \tilde{P}(x)\exp(\alpha x)$ avec $\deg \tilde{P} = 2$, $\alpha = 2$

(1) Primitives de $\tilde{P}(x)e^{\alpha x}$ est de la forme $P(x)e^{\alpha x}$ avec $\deg P = \deg \tilde{P}$

dans l'exple:

$P(x)e^{2x} = (ax^2 + bx + c)e^{2x}$

$[(ax^2 + bx + c)e^{2x}]' = (2ax^2 + 2(a+b) + 2c + b)e^{2x} \overset{!}{=} x^2 \cdot e^{2x}$

$2ax$

alors $\iff 2aX^2 + 2(a+b)X + 2c + b \overset{!}{=} X^2 + 0\cdot X + 0\cdot 1$

$\iff \begin{cases} 2a = 1 \\ a + b = 0 \\ 2c + b = 0 \end{cases}$ $\quad a = \frac{1}{2}$, $b = -\frac{1}{2}$, $c = \frac{1}{4}$

$\implies \int x^2 e^{2x}\,dx = \left(\frac{1}{2}X^2 - \frac{1}{2}X + \frac{1}{4}\right)e^{2x} + \underline{C}$

"à mon avis, un peu plus rapide que IPP ici"

2) $\int \sin(2x)e^{3x}\,dx$ $\quad (e^{3x})' = 3\cdot e^{3x}$

la méthode: $\left(\left[A\cdot\sin(2x) + B\cos(2x)\right]e^{3x}\right)' = \left[(3A-2B)\sin(2x) + (3B+2A)\cos(2x)\right]e^{3x} \overset{!}{=} \sin(2x)e^{3x}$

$(\sin 2x)' = 2\cos(2x)$, $(\cos 2x)' = -\sin\cdot 2x$, $-2B$

$\Longleftrightarrow 3A-2B=1$ et $2A+3B=0 \Longleftrightarrow \begin{pmatrix}3 & -2\\ 2 & 3\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \Longleftrightarrow \begin{pmatrix}A\\B\end{pmatrix} = \frac{1}{13}\begin{pmatrix}3 & 2\\ -2 & 3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{13}\begin{pmatrix}3\\-2\end{pmatrix}$

(2) Pour $\boxed{\sin(\beta x)\cdot e^{\alpha x}}$ et/ou $\cos(\beta x)\cdot e^{\alpha x}$ on cherche une solution de la forme:
$A\cdot\sin(\beta x)\cdot e^{\alpha x} + B\cdot\cos(\beta x)\cdot e^{\alpha x}$

III.4 Intégration de fractions rationelles

$\int \frac{P(x)}{Q(x)}\,dx$ $\quad$ **1er étape**: division euclidienne et D.E.S. du reste.

résultat:
$$\underbrace{\int \tilde{P}(x)\,dx}_{①} + \underbrace{\int \frac{A}{x-c_1}\,dx + \int \frac{\tilde{A}}{(x-c_1)^2}\,dx + \ldots}_{②} + \underbrace{\int \frac{B+Cx}{x^2+px+q}\,dx + \int \frac{\tilde{B}+\tilde{C}x}{(x^2+px+q)^2}\,dx}_{③}$$
$\Delta \equiv p^2-4q < 0$

2me étape:

pour ① et ②:
$$\int x^\alpha\,dx = \begin{cases} \ln(|x|)+C, & \text{si } \alpha=-1\\ \frac{x^{\alpha+1}}{\alpha+1}+C, & \text{si } \alpha\neq -1\end{cases}$$
(où définie)

$$\Longrightarrow ①\ \int \underbrace{(a_0+a_1x+a_2x^2+\ldots+a_nx^n)}_{\sum_{k=0}^n a_k\cdot x^k}dx = \sum_{k=0}^n a_k\cdot\int x^k\cdot dx = \left(\sum_{k=0}^n \frac{a_k}{k+1}\cdot x^{k+1}\right)+C$$

$n>1$ $(n\geq 2)$:
$$\Longleftrightarrow \int \frac{1}{x^n}\,dx = \int x^{-n}\,dx = \frac{x^{-n+1}}{-n+1}+C = \frac{1}{1-n}\cdot\frac{1}{x^{n-1}}+C \equiv -\frac{1}{n-1}\cdot\frac{1}{x^{n-1}}+C$$

$$\Longrightarrow ②\ \text{i)}\ \int \frac{1}{x-c}\,dx = \ln(|x-c|)+C$$
(par exple avec CdV $u=x-c$, $du=dx$)

ii) $n\geq 2$:
$$\int \frac{1}{(x-c)^n}\,dx = \frac{-1}{n-1}\,\frac{1}{(x-c)^{n-1}}+C$$

$\longrightarrow \int \frac{1}{x^n}\,dx = -\frac{1}{n-1}\cdot\frac{1}{x^{n-1}}+C$

$n=2$: $\int \frac{1}{x^2}\,dx = -\frac{1}{x}+C$

$n=3$: $\int \frac{1}{x^3}\,dx = -\frac{1}{2}\cdot\frac{1}{x^2}+C$

pour ③ il faut TRAVAILLER!

Stratégie: i) $\frac{1}{x^2+px+q}\,dx \xrightarrow{\text{CdV}} \int \frac{1}{1+x^2}\,dx$

Pour avoir de à,

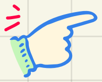
ii) $n\in\mathbb{N}^*$
$$\int \frac{x}{(x^2+px+q)^n}\,dx = \frac{1}{2}\int\frac{2x+p}{(x^2+px+q)^n}\,dx - \frac{p}{2}\int\frac{1}{(x^2+px+q)^n}\,dx$$

Rq: $(x^2+px+q)' = 2x+p$

⊕ $$\boxed{\int\frac{2x+p}{(x^2+px+q)^n}\,dx = \int\frac{\varphi'(x)}{\varphi(x)^n}\,dx \overset{\text{CdV}}{=} -\frac{1}{n-1}\cdot\frac{1}{\varphi(x)^{n-1}}+C}$$

$f(x) = \frac{1}{x^n}$, $\varphi(x) = x^2+px+q$

⊖ $$\int\frac{1}{(x^2+px+q)^n}\,dx \begin{cases}\text{pour } n=1 \rightarrow \text{i)}\\ n>1 \text{ plus tard.}\end{cases}$$

Exple: $I = \int_0^2 \frac{3}{x^3+1}\,dx$  Type exam

étape 1 : décomposer $\frac{3}{x^3+1}$ en éléments simples.

$$x^3+1 = (x+1)\underbrace{(x^2-x+1)}_{\Delta = 1-4 = -3}$$

Substitution :

$$\frac{3}{x^3+1} = \frac{A}{x+1} + \frac{B+Cx}{x^2-x+1} = \frac{1}{x+1} + \frac{2-x}{x^2-x+1}$$

$$\boxed{A} = \left(\frac{3}{\underbrace{x^2-x+1}_{1+1+1=3}}\right)\Bigg|_{x=-1} = \boxed{1}$$

$X=0$: $3 = A+B$

$\Rightarrow \boxed{B=2}$

$x=1$: $\frac{3}{2} = \frac{1}{2} + \frac{2+C}{1}$

$\Rightarrow \boxed{C=-1}$

$$I = \underbrace{\left[\ln(|x+1|)\right]_0^2}_{\ln(3)} - \frac{1}{2}\int_0^2 \frac{2x}{x^2-x+1}\,dx + 2\int_0^2 \frac{1}{x^2-x+1}\,dx = \ln(3) - \frac{1}{2}\int_0^2 \frac{2x-1}{x^2-x+1} + \frac{3}{2}\int_0^2 \frac{1}{x^2-x+1}\,dx$$

$\ln(2+1) - \cancel{\ln(0)}$

$(x^2-x+1)' = 2x-1$

$-1 \cdot -\frac{1}{2} = +\frac{1}{2}$

$2 - \frac{1}{2} = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$

$$\int \frac{2x-1}{x^2-x+1}\,dx = \ln\underbrace{(x^2-x+1)}_{>0} + C$$

⚠ $u = x^2-x+1$ pour $x \in [0,2]$ n'est pas une bijection $\longrightarrow$ CdV 1)

$$I = \underbrace{\ln(3) - \frac{1}{2}\left[\ln(x^2-x+1)\right]_0^2}_{\ln(3) - \frac{1}{2}(\ln(3))} + \frac{3}{2} \quad := J$$

$4-2+1=3$

$\frac{1}{2}\ln(3) = \ln(\sqrt{3})$

où $J := \int_0^2 \frac{1}{x^2-x+1}\,dx$

$$\boxed{I = \ln(\sqrt{3}) + \frac{3}{2}\cdot\frac{\sqrt{3}}{2}\cdot J}$$

$$J = \int_0^2 \frac{1}{x^2-x+1}\,dx = \int_{-1/2}^{3/2} \frac{1}{u^2+\frac{3}{4}}\,du = \frac{4}{3}\,\frac{\sqrt{3}}{2}\int_{-1/\sqrt{3}}^{\sqrt{3}} \frac{1}{v^2+1}\,dv$$

But effacer-le

$$x^2-x+1 = \underbrace{\left(x-\frac{1}{2}\right)^2}_{x^2-x+\frac{1}{4}} + \frac{3}{4}$$

CdV

① $u := x - \frac{1}{2}$ (une bijection !)

② $du = dx$

③

x	u
2	3/2
0	-1/2

Bornes

$u^2 + \frac{3}{4} = \frac{3}{4}\Big(\underbrace{\frac{4}{3}u^2}_{v^2}+1\Big)$

où $v = \frac{2}{\sqrt{3}}u$, $dv = \frac{2}{\sqrt{3}}du$

u	v
3/2	$\sqrt{3}$
-1/2	$-1/\sqrt{3}$

$$J = \frac{4}{3}\left[\arctan(v)\right]_{v=-\frac{1}{\sqrt{3}}}^{\sqrt{3}} = \frac{4}{3}\Big(\underbrace{\arctan(\sqrt{3})}_{\pi/3} + \underbrace{\arctan\left(\frac{1}{\sqrt{3}}\right)}_{\pi/6}\Big) = \frac{2\pi}{3}$$

$-\arctan\left(-\frac{1}{\sqrt{3}}\right) = +\arctan\left(\frac{1}{\sqrt{3}}\right)$ car arctan est fonct° impaire.

alors $J = \frac{2\pi}{3}$

et $\boxed{I = \ln(\sqrt{3}) + \frac{\sqrt{3}}{2}\pi}$