Sol Ex

Pendant le CM 9
Fiche de TD 3, Ex. 8

30/03
Alicia
ALGÈBRE 2

Ex. 8

$\Psi(P) = P(X+1) - P(X)$

1) Soient $P, Q \in \mathbb{R}_n[X]$ et $\lambda \in \mathbb{R}$
Alors, on a :
$\Psi(P+\lambda Q) = (P+\lambda Q)(X+1) - (P+\lambda Q)(X)$
$= P(X+1) + \lambda Q(X+1) - P(X) - \lambda Q(X)$
$= \Psi(P) + \lambda Q(Q)$
Donc, Ψ est bien une appl. lin.

2) $\text{Ker}(\Psi) = \{P \in \mathbb{R}_n[X] : \Psi(P) = 0\}$
$P \in \text{Ker}(\Psi)$
$\Psi(P) = 0 \iff P(X+1) - P(X) = 0$
$\Rightarrow P(X+1) = P(X)$
$P(X) = \sum_{k=0}^{n} a_k X^k$, $a_k \in \mathbb{R}$

$\Rightarrow \sum_{k=0}^{n} \underbrace{a_k (X+1)^k}_{A} = \sum_{k=0}^{n} a_k \cdot X^k = 0 \quad \forall x \in \mathbb{R}$

$= \sum_{k=0}^{n} a_k \left[\cancel{X^k} + \binom{k}{k-1} X^{k-1} + \dots + \binom{k}{1} X + \cancel{-X^k} \right]$
$+ a_0 - a_0 = 0$
$= \sum_{k=1}^{n} a_k \left[\sum_{l=0}^{k-1} X^l \right] = 0$

$\Rightarrow \sum_{k=1}^{n-1} d_k X^k \quad d_k \in \mathbb{R}$

$\Rightarrow a_k = 0$ si $k = 1, \dots, n-1$
$\dim(\text{Ker}(\Psi)) = 2$

$\Rightarrow P(X) = a_0$
$\Rightarrow \text{Ker}\,\Psi = \{P \in \mathbb{R}_n[X] : P(X) = a_0 \in \mathbb{R}\}$

3) Thm du Rang :
$\dim(\mathbb{R}_n[X]) = \dim(\text{Ker}\,\Psi) + \text{rg}(\Psi)$
$\Rightarrow n+1 = 2 + \text{rg}\,\Psi$
$\Rightarrow \text{rg}\,\Psi = n-1$

$\mathbb{R}_n[X] = \text{Vect}\{1, X, X^2, \dots, X^n\}$

4) $P(X) = \sum_{k=0}^{n} a_k \cdot X^k$

$\Psi(P) = P(X+1) - P(X)$
$= \sum_{k=0}^{n} a_k (X+1)^k - \sum_{k=0}^{n} a_k \cdot X^k$
$= \sum_{k=1}^{n} a_k \left[(X+1)^k - X^k \right] + \cancel{a_0} - \cancel{a_0}$
$= \sum_{k=1}^{n-1} c_k \cdot X^k$; $\text{Im}(\Psi) = \text{Vect}\{X, X^2, \dots, X^{n-1}\}$

5) $P \in \text{Ker}(\Psi)$, $Q \in \text{Im}(\Psi)$
$P(X) = a_n X^n + a_0$, $Q(X) = \sum_{k=1}^{n-1} X^k b_k$

$\Rightarrow P(X) + Q(X) = \sum_{k=1}^{n-1} c_k X^k$, $c_0 = a_0$, $c_n = a_n$ (2 coeff. réels)
$c_k = b_k$ sinon

$\Rightarrow \text{Ker}\,\Psi + \text{Im}\,\Psi \subseteq \mathbb{R}_n[X]$

$\sum_{k=0}^{n} (X+1)^k - X^k$
$= X^k + \binom{k}{k-1} X^{k-1} + \dots + 1 - X^k$

- $\text{Im}\,\Psi + \text{Ker}\,\Psi = \mathbb{R}[X]$

(pour la somme directe, il faut une intersection).
$P(X) + Q(X) = a_n X^n + a_0 + \sum_{k=1}^{n-1} b_k X^k$ on pose $(c_k) \in$ suite $\mathbb{R}$
$= \sum_{k=0}^{n} c_k X^k \to \mathbb{R}_n[X]$

$c_k = \begin{cases} a_0 & \text{si } k=0 \\ a_n & \text{si } k=n \\ b_k & \text{si non} \end{cases}$

$c_n X^n = a_n X^n$

$P \in \underbrace{\text{Ker}\,\Psi}_{a_n = 0 \text{ pour } k \neq 0, n} \cap \text{Im}\,\Psi$

$P(X) \sum_{k=0}^{n} a_k X^k \to a_0 = 0, a_n = 0$

$\Rightarrow \text{Im}\,\Psi \oplus \text{Ker}\,\Psi = \mathbb{R}_n[X]$