Sol Ex

Pendant le CM 8
Fiche de TD 3, Ex. 14

23/03
Gricey
ALGÈBRE 2

Exercice 14.

$f : \mathbb{R}^4 \to \mathbb{R}^3$, $x \mapsto A \cdot x$; $g : \mathbb{R}^3 \to \mathbb{R}^4$, $x \mapsto {}^tA \cdot x$ → transposé !

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{pmatrix}$$

① Ker f ?

$\{x \in \mathbb{R}^4,\ f(x) = 0\} = \{x \in \mathbb{R}^4,\ Ax = 0\}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \ker(f) \iff \begin{cases} x_1 + 2x_2 + x_3 - x_4 = 0 \\ 9x_1 + 5x_2 + 2x_3 + 2x_4 = 0 \\ 7x_1 + x_2 + 4x_4 = 0 \end{cases}$$

$$\iff \begin{cases} x_1 + 2x_2 + x_3 - x_4 = 0 \\ -13x_2 + 7x_3 + 11x_4 = 0 \\ -13x_2 - 7x_3 + 11x_4 = 0 \end{cases}$$

$$\iff \begin{cases} x_1 + 2x_2 = -x_3 + x_4 \\ -13x_2 = 7x_3 - 11x_4 \end{cases}$$

$$\iff \begin{cases} x_1 = \frac{14}{13}x_3 - \frac{22}{13}x_4 - x_3 + x_4 \\ x_2 = -\frac{7}{13}x_3 + \frac{11}{13}x_4 \end{cases}$$

$$\iff \begin{cases} x_1 = \frac{1}{13}x_3 - \frac{9}{13}x_4 \\ x_2 = -\frac{7}{13}x_3 + \frac{11}{13}x_4 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/13 \\ -7/13 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -9/13 \\ 11/13 \\ 0 \\ 1 \end{pmatrix}$$

$$\ker(f) = \text{Vect}\left(\begin{pmatrix} 1 \\ -7 \\ 13 \\ 0 \end{pmatrix}, \begin{pmatrix} -9 \\ 11 \\ 0 \\ 13 \end{pmatrix}\right)$$

"le rang est la dim de im"

② $\text{Rg}(f) = \dim(\text{Im}(f))$

↳ "Gauss sur les colonnes"

Rappel : $\text{Im}(f) = \text{vect}(\text{colonnes de } A)$

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \\ 7 \end{pmatrix} x_1 + \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} x_2 + \dots$$

Thm : $\text{rg}(f) = 4 - \dim(\ker(f))$

↑ dim espace de départ

$= 4 - 2 = 2$

Sinon.

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 9 & -13 & -7 & 11 \\ 7 & -13 & -7 & 11 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 9 & -13 & 0 & 0 \\ 7 & -13 & 0 & 0 \end{pmatrix}$$

$\text{rg} = 2$; nb de colonnes non nulles.

matrice échelonné en colonnes !

③ Si base de $\text{Im}(f)$

$$\text{Im}(f) = \text{vect}\left(\begin{pmatrix} 1 \\ 9 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ -13 \\ 13 \end{pmatrix}\right) = \text{vect}\left(\begin{pmatrix} 1 \\ 9 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}\right)$$

④ et ⑤

$\text{rg}(g) = 2$

$$A^T = \begin{pmatrix} 1 & 9 & 7 \\ 2 & 5 & 1 \\ 1 & 2 & 0 \\ -1 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ 2 & -13 & -13 \\ 1 & -7 & -7 \\ -1 & 11 & 11 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 1 & -13 & 0 \\ 1 & -7 & 0 \\ -1 & 11 & 0 \end{pmatrix}$$

↳ Base de $\text{im}(g)$.

⑥ $\ker(g) = ?$

$\{x \in \mathbb{R}^3,\ g(x) = 0\}$

$Ax = 0$; ${}^t(A \cdot x) = 0$; ${}^t({}^tA \cdot x) = 0$; ${}^tx \cdot A = 0$

Formules : ${}^t(A \cdot B) = {}^tB \cdot {}^tA$; ${}^t({}^tA) = A$

$$\begin{pmatrix} 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} & A & \cdot \\ {}^tx & = & \end{pmatrix}$$

produit scalaire entre les deux

$x \in \text{Im}(f)^{\perp}$ (orthogonal) $= \text{vect}\left(\begin{pmatrix} 1 \\ 9 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}\right)^{\perp}$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ vérifie $\begin{cases} x_1 + 9x_2 + 7x_3 = 0 \\ 2x_1 + 5x_2 + x_3 = 0 \end{cases}$ | 2 eq's.

$$\iff \begin{cases} x_1 + 9x_2 + 7x_3 = 0 \\ 0 - 13x_2 - 13x_3 = 0 \leftarrow L_2 - 2L_1 \end{cases}$$

$$\iff \begin{cases} x_1 + 9x_2 = -7x_3 \\ x_2 = -x_3 \end{cases}$$

$$\iff \begin{cases} x_1 = -9x_2 - 7x_3 \\ x_2 = -x_3 \end{cases}$$

$$\iff \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$\ker(g) = \text{Vect}\left(\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right)$

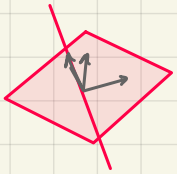
$\text{Im}(f) = \text{Vect}\left(\begin{pmatrix} 1 \\ 9 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}\right)$

⑦ Non car $\text{Im}(f) + \ker(g)$

Comment m.q. sont en somme directe ?

↓

si c'est le cas on concatène ça forme 1 fois une base de $\mathbb{R}^3$



$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 9 \\ 7 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

Det ≠ 0

Déterminant ⇝ $(18 + 5 - 14) - (\cancel{8} - 70 - 1) \neq 0$

⇝ $\ker(g) \oplus \text{Im}(f) = \mathbb{R}^3$

Ccl : on est en somme directe.