Sol Ex

Pendant le CM 11
~~Fiche de TD ___, Ex.~~

20/04
Briay
ALGÈBRE 2

1) $f : \mathbb{R}_2[X] \to \mathbb{R}_2[X]$,
$P \mapsto (P'' + P' + P(0)X)$

a) M.q. $f \in End(\mathbb{R}_2[X])$ b) Calculer $A = [f]_{\mathcal{C}\mathcal{C}}$, $\mathcal{C} = (1, X, X^2)$ c) Déterminer $\ker(A)$ et $\text{im}(A)$
d) M.q. $\ker A \cap \text{im } A = \{\begin{pmatrix}0\\0\\0\end{pmatrix}\}$ e) Déterminer $\ker f$ et $\text{im } f$. f) Calculer $B = [f]_{\mathcal{B}\mathcal{B}}$ pour $\mathcal{B} = (X, 1, X^2 - 2X - 2)$

a) $\forall P, Q \in \mathbb{R}_2[X]$, $\forall \lambda \in \mathbb{R}$.
$f(P + \lambda Q) \overset{\text{def}}{=} 2\left[(P+\lambda Q)'' + (P+\lambda Q)' + (P+\lambda Q)(0)\cdot X\right] =$
$= 2\left[P'' + \lambda \cdot Q'' + P' + \lambda Q' + P(0)\cdot X + \lambda(Q)\cdot X\right] =$
$= \underbrace{2(P'' + P' + P(0)\cdot X)}_{f(P)} + \lambda \cdot \underbrace{2(Q'' + Q' + Q(0)\cdot X)}_{f(Q)}$
$\Rightarrow P(P + \lambda Q) = f(P) + \lambda \cdot f(Q)$ □.

le 1° élément de la base

b) $f(1) = 2(0 + 0 + 1\cdot X) = 2X = \underline{0}\cdot 1 + \underline{2}\cdot X + \underline{0}\cdot X^2$
$f(X) = 2(0 + 1 + 0\cdot X) = \underline{2}\cdot 1 + \underline{0}\cdot X + \underline{0}\cdot X^2$
$f(X^2) = 2(2 + 2X + 0\cdot X) = \underline{4}\cdot 1 + \underline{4}\cdot X + \underline{0}\cdot X^2$

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

c) $\ker A$:
$$\left(\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right) \quad \begin{matrix} \Rightarrow & x_1 = -2\lambda \\ \Rightarrow & 2x_2 + 4\lambda = 0 \Rightarrow \end{matrix}$$
$x_3 = \lambda$
$\Rightarrow \text{rg } A = 2$, $\text{rg } A \overset{\text{def}}{=} \dim(\text{im } A) \Rightarrow \text{im } A = \text{Vect}(e_1, e_2) \equiv \text{Vect}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right)$

$\boxed{\ker A = \text{Vect}\left(\begin{pmatrix}-2\\-2\\1\end{pmatrix}\right)}$

$\boxed{\text{im } A = \text{Vect}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right)}$

d) $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \in \ker A \setminus \{\begin{pmatrix}0\\0\\0\end{pmatrix}\} \Rightarrow x_3 \neq 0 \Rightarrow \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \notin \text{im}(A) \Rightarrow \ker A \cap \text{im } A = \{\begin{pmatrix}0\\0\\0\end{pmatrix}\}$

Rq: $\ker A + \text{im } A = \ker A \oplus \text{im } A = \mathbb{R}^3$ ✳

e) Pour la base canonique $\mathcal{C}$ on a

$\mathbb{R}^3 \ni \begin{pmatrix}a\\b\\c\end{pmatrix} \overset{\cdot(-2),\ \cdot(-2),\ \cdot 1}{\longleftrightarrow} a\cdot 1 + b\cdot X + c\cdot X^2 \in \mathbb{R}_2[X]$
$-2\cdot 1 + -2\cdot X + 1\cdot X^2$

$\mathbb{R}^3 \ni \underbrace{\ker A}_{\text{input}} = \underbrace{\text{Vect}\left(\begin{pmatrix}-2\\-2\\1\end{pmatrix}\right)}_{\text{at this point}} \overset{\sim}{\longleftrightarrow} \text{Vect}(-2 - 2X + X^2) = \underbrace{\ker f \in \mathbb{R}_2[X]}_{\text{output}}$

$\mathbb{R}^3 \ni \underbrace{\text{im } A}_{\text{input}} = \underbrace{\text{Vect}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right)}_{\text{at this point}} \overset{\sim}{\longleftrightarrow} \text{Vect}(1, X) = \underbrace{\text{im } f \in \mathbb{R}_2[X]}_{\text{output}}$

f) $\mathcal{B} = (X, 1, X^2 - 2X - 2)$
$f(X) = 2$ cf. ci-dessus.
$f(1) = 2X$
$f(X^2 - 2X - 2) = 0$ ($\in \ker f$)

$\boxed{B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$

(Rq: $\ker f + \text{im } f = \mathbb{R}_2[X]$) ✳

À vous !

2) $f: \mathbb{R}^6 \to \mathbb{R}^3$, $x \mapsto A \cdot x$, $A = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 4 & -1 & 2 & -3 & 0 \end{pmatrix}$

a) Trouver $S_i = \{x \in \mathbb{R}^6 \mid f(x) = b_i\}$ pour $i = 1,2,3$

et $b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $b_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $b_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

b) Base Ker f et im f

Résultats pr vérifier.

$S_1 = \{e_1 + u,\ u \in \ker f\}$

$S_2 = \varnothing$

$S_3 = \ker f = \text{Vect}\left(\begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)$ → tjs une base libre.

3) Ex. III.10

$f: \mathcal{M}_2 \to \mathcal{M}_2$, $A \mapsto \frac{1}{2}(A - A^T)$

1) M.q. $f \in \text{End}(\mathcal{M}_2) : \forall A, B \in \mathcal{M}_2,\ \forall \lambda \in \mathbb{R}$:

$f(A+B) = \dots = f(A) + \lambda \cdot f(B)$ (à vous)

2) base de Ker f et im f

3) • Ker f + im f = ?

• $f^2 = f$?

$\boxed{\ker f = \{A \in \mathcal{M}_2 \mid A = A^T\}}$

$\text{im } f = \{B \in \mathcal{M}_2 \mid B = \frac{1}{2}(A - A^T)\}$

$B \in \text{im } f \iff B^T = -B$ $\quad (\Rightarrow B = f(B))$

$\Rightarrow \boxed{\text{im } f = \{A \in \mathcal{M}_2 \mid A^T = -A\}}$

$A \in \text{im } f \Rightarrow A = \begin{pmatrix} 0 & +a \\ -a & 0 \end{pmatrix} \Rightarrow \text{im } f = \text{Vect}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$

$A \in \ker f = \begin{pmatrix} a & b \\ +b & d \end{pmatrix} \Rightarrow \ker f = \text{Vect}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$

3) • $A \in \ker f \cap \text{im } f \Rightarrow \boxed{\begin{matrix} A^T = A \\ A^T = -A \end{matrix}}$ car $A \in \ker f$, $A \in \text{im } f$

$\hookrightarrow A = -A \Rightarrow A = 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

• $f^2(A) = f(f(A)) = f\left(\frac{1}{2}(A - A^T)\right) = \frac{1}{2}f(A) - \frac{1}{2}f(A^T) = \frac{1}{4}f(A - A^T) - \frac{1}{4}(A^T - A) =$

$= \frac{1}{2}(A - A^T) = \boxed{f(A)} \Rightarrow f^2 = f$

Rq: Alors f est une projection. notamment un projecteur sur les matrices anti-sym. parallèlement aux matrices sym.