Sol Ex

Pendant le CM 8
Fiche de TD 4, Ex. 30, 31

21/03
Gricey
ANALYSE 2

Évaluer l'intégrale :

$$I := \int_0^{\pi/2} \frac{1}{\underline{1+\sin x+\cos x}}\,dx$$

CdV

$$u = \tan\left(\frac{x}{2}\right), \quad x = 2\arctan(u) \qquad dx = \frac{2}{1+u^2}\,du \qquad \cos(x) = \frac{1-u^2}{1+u^2} \qquad \sin(x) = \frac{2u}{1+u^2}$$

Bornes

$$\tan\left(\frac{0}{2}\right) = 0 \qquad \tan\left(\frac{\pi}{4}\right) = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

$$1+\sin(x)+\cos(x) = 1 + \frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2} = \frac{1+\cancel{u^2}+2u+1-\cancel{u^2}}{1+u^2} = \frac{2u+2}{1+u^2}$$

$\sin(x) = \sin(2\,\mathrm{Arctan}(u))$
$= \sin(\mathrm{Arctan}(u))\cos(\mathrm{Arctan}(u))$

On a
$$I = \int_0^1 \frac{\frac{2}{1+u^2}}{\frac{2u+2}{1+u^2}}\,du = \int_0^1 \frac{2}{2u+2}\,du = \int_0^1 \frac{1}{1+u}\,du = \Big[\ln|1+u|\Big]_0^1 = \ln(2)$$

□.

Ex 31

$$I := \int_0^{\pi/3} \frac{3\cdot\sin(x)}{3\cos(x)-1}$$

$$\tan(0) = 0, \quad \tan\frac{\pi}{6} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

Bornes

Méthode 1 : J'ai rien vu

$$u = \tan\left(\frac{x}{2}\right); \quad dx = \frac{2}{1+u^2}\,du$$

$$\cos(x) = \frac{1-u^2}{1+u^2} \qquad \sin(x) = \frac{2u}{1+u^2}$$

$$3\cos(x) - 1 = \frac{3-3u^2}{1+u^2} - \left(\frac{1+u^2}{1+u^2}\right) = \frac{2-4u^2}{1+u^2}$$

$$3\sin(x)\,dx = \frac{6u}{1+u^2} \times \frac{2}{1+u^2}\,du = \frac{12u}{(1+u^2)^2}\,du$$

$$J = \int_0^{1/\sqrt{3}} \frac{\frac{12u}{(1+u^2)^2}}{\frac{2-4u^2}{1+u^2}}\,du = \int_0^{1/\sqrt{3}} \frac{12u}{(1+u^2)(2-4u^2)}\,du = \int_0^{1/\sqrt{3}} \frac{6u}{(1+u^2)(1-\sqrt{2}u)(1+\sqrt{2})}\,du$$

$\div 2$

$$F(x) = \frac{6x}{(1+x^2)(1-x\sqrt{2})(1+x\sqrt{2})} \qquad \deg F < 0 \qquad \exists (a,b,c,d) \in \mathbb{R}^n, \quad F(x) = \frac{ax+b}{1+x^2} + \frac{c}{1+x\sqrt{2}} + \frac{d}{1-x\sqrt{2}}$$

$$(1+x\sqrt{2})F(x) = \frac{6x}{(1+x^2)(1-x\sqrt{2})} = c + (1+x\sqrt{2})\left(\frac{ax+b}{1+x^2} + \frac{d}{1-x\sqrt{3}}\right)$$

Pour $x = \frac{-\sqrt{2}}{2}$, $\frac{-6\sqrt{2}}{2\left(\frac{3}{2}\right)(2)} = \frac{-6\sqrt{2}}{6} = -\sqrt{2} = c$

Pour $x = \frac{\sqrt{2}}{2}$, $\sqrt{2} = d$

$$* \quad F(x) = \frac{ax+b}{1+x^2} - \frac{\sqrt{2}}{1+x^2} + \frac{\sqrt{2}}{1-x^2}$$

$$* \quad F(0) = 0 = \frac{b}{1} - \sqrt{2} + \sqrt{2} = b$$

$$F(x) = \frac{ax}{1+x^2} - \frac{\sqrt{2}}{1+x^2} + \frac{\sqrt{2}}{1-x^2}$$

$$(1+x^2)F(x) = \frac{6x}{(1-x\sqrt{2})(1+x\sqrt{2})} = ax + (1+x^2)\,(*)$$

$$x = i, \quad \frac{6i}{(1-i\sqrt{2})(1+i\sqrt{2})} = \frac{6i}{3} = ai, \text{ alors } a = 2$$

$$F(x) = \frac{2x}{(1+x^2)} + \frac{\sqrt{2}}{(1-x\sqrt{2})} - \frac{\sqrt{2}}{(1+x\sqrt{2})}$$

$$J = \int_0^{1/\sqrt{3}} \frac{2u}{1+u^2}\,du + \int_0^{1/\sqrt{3}} \frac{\sqrt{2}}{1-u\sqrt{2}}\,du - \int_0^{1/\sqrt{3}} \frac{\sqrt{2}}{1+u\sqrt{2}}\,du$$

$$= \Big[\ln|1+u^2|\Big]_0^{1/\sqrt{3}} + \Big[-\ln|1-u\sqrt{2}|\Big]_0^{1/\sqrt{3}} - \Big[\ln|1+u\sqrt{2}|\Big]_0^{1/\sqrt{3}}$$

$\ln a + \ln b = \ln(ab)$
$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

$$= \ln\left(1+\frac{1}{3}\right) - \ln\left(1-\sqrt{\frac{2}{3}}\right) - \ln\left(1+\sqrt{\frac{2}{3}}\right)$$

$$= \ln\left(\frac{\frac{4}{3}}{\left(1-\sqrt{\frac{2}{3}}\right)\left(1+\sqrt{\frac{2}{3}}\right)}\right) = \ln\left(\frac{\frac{4}{3}}{1-\frac{2}{3}}\right) = \ln\left(\frac{\frac{4}{3}}{\frac{1}{3}}\right) = \ln(4)$$

□.

Méthode 2 : J'ai vu

$\cos(x)' = -\sin(x)$

$$I := \int_0^{\pi/3} \frac{3\cdot\sin(x)}{3\cos(x)-1} = \frac{-3\cos(x)}{3\cos(x)-1} \longleftrightarrow \frac{u'}{v}$$

$\Updownarrow$

$u = \frac{u'}{u}$

$(3\cdot\cos(x)-1)' = -3\sin(x)$

$\int \tan(x)$

$= \int \frac{\sin x}{\cos x} = -\ln|\cos x| + C$

$$J = \Big[-\ln|3\cos(x)-1|\Big]_0^{\pi/3}$$

$$= -\ln\left(\frac{3}{2}-1\right) + \ln(2) = \ln(2) - \ln\left(\frac{1}{2}\right) = \ln\left(\frac{\frac{2}{1}}{\frac{1}{2}}\right) = \ln(4)$$