SolEx

Pendant le CM 7
Fiche de TD 4, Ex. 12 (continuation) & Ex. 23

14/03
ANALYSE 2

Ex. 12

1) $C(\alpha) := \int_0^{\alpha} \cos^2 x \, dx$

• Résultats obtenus de 1) et 2) (cf SolEx_CM7)

$$\boxed{C(x) = \frac{\sin x \cdot \cos x + x}{2}} \quad \left(= \frac{x}{2} + \frac{\sin(2x)}{4} \right)$$

2) $S(x) := \int_0^{x} \underbrace{\sin^2 t}_{= 1-\cos^2 t} \, dt$

$$\boxed{S(x) = \frac{-\sin x \cdot \cos x + x}{2}}$$

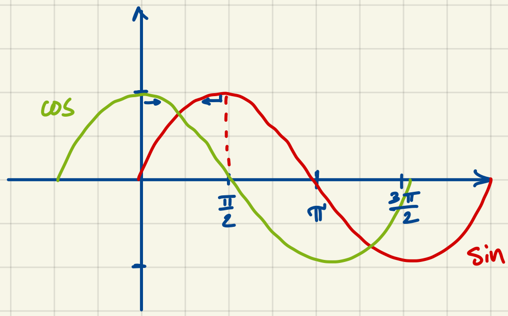
3) On pose : $W_n := \int_0^{\pi/2} \sin^n(x)\, dx$ l'integral de Wallis.

↳ M.q. $W_n = \int_0^{\pi/2} \cos^n(x)\, dx$

$$\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$$

$$\Longleftrightarrow \cos x = \sin\left(x + \frac{\pi}{2}\right)$$

$$\overset{x \mapsto -x}{\Longleftrightarrow} \boxed{\cos x = \sin\left(\frac{\pi}{2} - x\right)}$$



CdV : $\boxed{x := \frac{\pi}{2} - u}$ $\Longrightarrow \sin(x) = \sin\left(\frac{\pi}{2} - u\right) = \cos(u)$

$dx = -du$

$u = \frac{\pi}{2} - x$

x	u
$\frac{\pi}{2}$	0
0	$\frac{\pi}{2}$

$$\left(\begin{array}{l} x = \varphi(u) \\ dx = \varphi'(u)\, du \end{array} \right)$$

$$\underbrace{W = \int_0^{\pi/2} \sin^n(x) \cdot dx}_{\text{Wallis}} = -\int_{\pi/2}^{0} \cos^n(u)\, du = \boxed{\int_0^{\pi/2} \cos^n(u) \cdot du} \quad \square.$$

$$\left(\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx \right)$$

4) soit $n \geq 2$.

$$W_n = \int_0^{\pi/2} \sin^n x \cdot \sin^{n-1}(x) \cdot dx = \left[-\cos x \cdot \sin^{n-1}(x) \right]_0^{\pi/2} + \int_0^{\pi/2} \underbrace{\cos x^2}_{1-\sin^2 x} \cdot (n-1) \cdot \sin^{n-2}(x) \cdot dx$$

$u'(x) = \sin x$, $v(x) = \sin^{n-1}(x)$

$u = -\cos x$, $v' = (n-1) \cdot \sin^{n-2}(x) \cdot \cos(x)$

$$(n-1) \cdot \underbrace{\int_0^{\pi/2} \sin^{n-2}(x)\, dx}_{W_{n-2}} - (n-1) \underbrace{\int_0^{\pi/2} \sin^n(x)\, dx}_{W_n}$$

$$\Longrightarrow \cancel{W_n} = (n-1) \cdot W_{n-2} + (\cancel{1}-n)\, W_n \Longleftrightarrow \boxed{W_n = \frac{n-1}{n} \cdot W_{n-2}}$$

5) $W_n = \frac{n-1}{n} \cdot W_{n-2} = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot W_{n-4}$

(si $n \geq 4$)

Déf : (si $n \in \mathbb{N}^*$), $n!! = \begin{cases} n(n-2)\cdot(n-4) \cdots 2 & \text{, si } n \text{ paire} \\ n(n-2) \cdots 3 \cdot 1 & \text{, si } n \text{ impaire} \end{cases}$

$0!! := 1$

$(-1)!! := 1$

Rq :

$\boxed{(2N)!!}$ (càd n paire) $= 2N \underbrace{(2N-2)}_{2(N-1)} \underbrace{(2N-4)}_{2(N-2)} \cdots 4 \cdot 2 = \boxed{2^N N!}$

$\boxed{(2N-1)!!}$ (càd n impaire) $= (2N-1)(2N-3) \cdots = \frac{(2N)!}{(2N)!!} = \boxed{\frac{(2N)!}{2^N \cdot N!}}$

$$\boxed{W_n} = \frac{n-1}{n} \cdot W_{n-2} = \frac{(n-1)(n-3)}{n(n-2)} \cdot W_{n-4} = \dots = \boxed{\begin{cases} \frac{(n-1)!!}{n!!} \cdot W_0 \ , \ n \text{ paire} \\ \frac{(n-1)!!}{n!!} \cdot W_1 \ , \ n \text{ impaire} \end{cases}}$$

$$\boxed{W_0} \equiv \int_0^{\pi/2} \underbrace{\sin^0(x)}_{1} \cdot dx = \boxed{\frac{\pi}{2}}$$

$$\boxed{W_1} \equiv \int_0^{\pi/2} \sin(x)\, dx = \boxed{\left[-\cos x\right]_0^{\pi/2} = 1}$$

$$\boxed{\begin{array}{lll} W_n = \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2} & \text{si} & n \text{ paire} \\ W_n = \frac{(n-1)}{n!!} & \text{si} & n \text{ impaire} \end{array}}$$

$$\text{⊛} \Rightarrow \boxed{W_2 = \frac{1!!}{2!!} \cdot \frac{\pi}{2} = \frac{\pi}{4}}$$

$$\Rightarrow \boxed{W_3 = \frac{2!!}{3!!} \cdot \cancel{W_1}^{1} = \frac{2}{3}}$$

cf 2) $W_2 = S\left(\frac{\pi}{2}\right) = \frac{\frac{\pi}{2} - \sin\left(\frac{\pi}{2}\right)\cdot\cos\left(\frac{\pi}{2}\right)}{2} = \frac{\pi}{4}$ ✓

$= C\left(\frac{\pi}{2}\right) = \frac{\frac{\pi}{2} + \sin\left(\frac{\pi}{2}\right)\cdot\cos\left(\frac{\pi}{2}\right)}{2} = \frac{\pi}{4}$ ✓

Vérification

Ex.23 Calculer :

$$I = \underbrace{\frac{x+1}{(x^2+1)^2}}_{\text{c'est déjà un élém}^t\text{ simple !}} dx = \underbrace{\frac{1}{2}\int_2^3 \frac{2x}{(1+x^2)^2} dx}_{\text{Simple} \to \text{CdV : } u = 1+x^2} + \underbrace{\int_2^3 \frac{1}{(1+x^2)^2} dx}_{=: J} = \frac{1}{2}\underbrace{\int_5^{10} \frac{1}{u^2} du}_{\left[-\frac{1}{u}\right]_5^{10}} + J \Rightarrow \boxed{I = \frac{1}{2}\left(-\frac{1}{10} + \frac{1}{5}\right) + J = \frac{1}{20} + J}$$

$du = 2x\, dx$

Bornes

x	u
3	10
2	5

• Trois méthodes pour calculer $J \equiv \int_2^3 \frac{1}{(1+x^2)^2} dx$

(1) Méthode Standard

$$J = \int_2^3 \frac{1+x^2-x^2}{(1+x^2)^2} dx = \int_2^3 \frac{1}{1+x^2} dx = \underbrace{\int_2^3 \frac{1}{1+x^2} dx}_{[\arctan]_2^3} - \int_2^3 \frac{x^2}{(1+x^2)^2} dx = \arctan(3) - \arctan(2) - K$$

$$K := \int_2^3 \frac{x^2}{(1+x^2)^2} dx$$

$$K = \frac{1}{2}\int_2^3 x \cdot \frac{2x}{(1+x^2)^2} dx = \left[\frac{1}{2} \cdot \frac{-x}{1+x^2}\right]_2^3 + \frac{1}{2}\int_2^3 \frac{1}{1+x^2} dx = \frac{1}{2}\left(-\frac{3}{10} + \frac{2}{5}\right) + \frac{1}{2}\left(\arctan 3 - \arctan 2\right)$$

$u = x \qquad v' = \frac{2x}{(1+x^2)^2} = \frac{\varphi'(x)}{(\varphi(x))^2}$

$u' = 1 \qquad v = -\frac{1}{1+x^2}$

Rappel :
$$\int \frac{1}{x^2} dx = -\frac{1}{x} + C$$

$$\Rightarrow J = -\frac{1}{20} + \frac{1}{2}\left(\arctan 3 - \arctan 2\right)$$

$$\Rightarrow \boxed{I = \frac{1}{20} + J = \frac{1}{2}\left(\arctan 3 - \arctan 2\right)}$$ ← le résultat du Méthode 1.

• Pour simplifier le résultat :

$\arctan(3) - \arctan(2) \equiv \arctan(3) + \arctan(-2) =$

$3\cdot(-2) < 1$ $\overset{\uparrow}{=}$ (plus tard la Preuve) $\arctan\left(\frac{3+(-2)}{1-3\cdot(-2)}\right) = \arctan\left(\frac{1}{7}\right)$ alors $\boxed{I = \frac{1}{2}\arctan\left(\frac{1}{7}\right)}$ le résultat

(2) $J = \int \frac{1}{2x}\cdot\frac{2x}{(1+x^2)^2}\,dx = \underbrace{\left[\frac{1}{-2x(1+x^2)}\right]_2^3}_{\underbrace{-\frac{1}{60}+\frac{1}{20}}_{\frac{1}{30}}} - \underbrace{\int_2^3 \frac{1}{2x^2}\cdot\frac{1}{1+x^2}\,dx}_{\text{Pas un élément simple}}$

$u = \frac{1}{2x}$ $\quad v' = \frac{2x}{(1+x^2)^2}$

$u' = -\frac{1}{2x^2}$ $\quad v = -\frac{1}{1+x^2}$

$$\frac{1}{2x^2(1+x^2)} = \frac{A}{x} + \frac{\tilde{A}}{x^2} + \frac{B+Cx}{1+x^2}$$

$\Longrightarrow \tilde{A} = \frac{1}{2},\ \left[\frac{1}{2x^2}\right]_{x=i} = B + Ci$

$\Longrightarrow B = -\frac{1}{2},\ C = 0$

pour A : $x = 1$

$\frac{1}{4} = A + \underbrace{\frac{1}{2}}_{\tilde{A}} - \frac{1}{2}\cdot\frac{1}{2} \Longrightarrow \underline{A = 0}$

$$\boxed{\frac{1}{2}\cdot\frac{1}{x^2(1+x^2)} = \frac{1}{2}\left(\frac{1}{x^2} - \frac{1}{1+x^2}\right)}$$

On aurait pu deviner ceci!

$$\boxed{J} = \frac{1}{30} - \frac{1}{2}\int_2^3\left(\frac{1}{x^2} - \frac{1}{1+x^2}\right)dx = \frac{1}{30} - \frac{1}{2}\left[-\frac{1}{x} - \arctan x\right]_2^3 = \underbrace{\underbrace{\frac{1}{30}+\frac{1}{6}}_{\frac{1}{5}} - \frac{1}{4}}_{-\frac{1}{20}} + \frac{1}{2}\left(\arctan(3) - \arctan(2)\right)$$

$$\boxed{= -\frac{1}{20} + \frac{1}{2}\left(\arctan(3) - \arctan(2)\right)}$$

(3) $J = \int_2^3 \frac{1}{(1+x^2)^2}\,dx = \int_{\arctan 2}^{\arctan 3} \cos^4(u)\,\frac{1}{\cos^2(u)}\cdot du = \int_{\arctan 2}^{\arctan 3} \cos^2(u)\,du \overset{\text{Ex.12}}{=} C(\arctan(3)) - C(\arctan(2)) =$

$C(x) = \frac{x + \sin x\cdot\cos x}{2} + c$

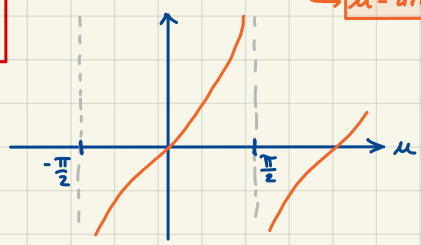
Rappel: $(\tan x)' = \frac{1}{\cos^2 x} \equiv 1 + \tan^2 x$

CdV : $\boxed{x = \tan u}$

$$\boxed{\frac{1}{(1+x^2)^2}} = \frac{1}{(1+\tan^2 u)^2} = \boxed{\cos^4(u)}$$

$u \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\rightarrow$ bijection

$\hookrightarrow \boxed{u = \arctan(x)}$



$dx = \frac{1}{\cos^2 u}\,du$

x	du
3	$\arctan(3)$
2	$\arctan(2)$

Bornes!

$= \frac{1}{2}\left(\arctan(3) - \arctan(2)\right)$

$+ \frac{1}{4}\left[\sin(2\arctan 3) - \sin(2\arctan 3)\right]$

$\sin x \cos x = \frac{\sin(2x)}{2}$

Un résultat possible, pourquoi le même?

$(\sin 2\theta) = \frac{2\tan\theta}{1+\tan^2\theta}$ (Preuve la prochaine fois "Weierstrass")

$\sin(2\arctan(a)) = \frac{2\tan(\arctan(a))}{1+(\tan(\arctan(a)))^2} = \frac{2a}{1+a^2}$

$$\boxed{J = \frac{1}{2}\underbrace{\left(\arctan(3) - \arctan(2)\right)}_{\arctan\left(\frac{1}{7}\right)} + \underbrace{\frac{1}{4}\left[\underbrace{\frac{2\cdot 3}{1+3^2}}_{\frac{3}{5}} - \underbrace{\frac{2\cdot 2}{1+2^2}}_{\frac{4}{5}}\right]}_{-\frac{1}{20}}}$$

alors le m̂ résultat!