7/03
Géricey
ANALYSE 2

3) $\boxed{I = \int_0^1 \underbrace{x}_{u'} \underbrace{\arctan(x)}_{v}\, dx} = \left[\frac{x^2}{2}\arctan(x)\right]_0^1 - \frac{1}{2}\int_0^1 x^2 \cdot \frac{1}{1+x^2}\, dx =$

$u' = x \qquad v = \arctan(x)$

$u = \frac{x^2}{2} \qquad v' = \frac{1}{x^2+1}$

$= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\underbrace{\int_0^1 \frac{x^2+1-1}{x^2+1}\, dx}_{\int_0^1 1\, dx - \int_0^1 \frac{1}{1+x^2}\, dx} = \frac{\pi}{8} - \frac{1}{2}\left(1-\frac{\pi}{4}\right) = \boxed{\frac{\pi}{4} - \frac{1}{2}}$

Une Autre Méthode : (plus rapide)

3) $\boxed{I = \int_0^1 x \underbrace{\arctan(x)}_{v}\, dx} = \left[\frac{x^2+1}{2}\arctan(x)\right]_0^1 - \frac{1}{2}\underbrace{\int_0^1 (x^2+1)\cdot\frac{1}{1+x^2}\, dx}_{1} =$

$u = \frac{x^2+1}{2} \qquad v = \arctan(x)$

$\frac{\pi}{4} \quad - \frac{1}{2} \cdot 1 \qquad = \boxed{\frac{\pi}{4} - \frac{1}{2}}$

4) $\int \sin^2 x\, dx = \int \underbrace{\sin x}_{u} \cdot \underbrace{\sin x}_{v'}\, dx$

$u = \sin x \qquad v' = \sin x$

$u' = \cos x \qquad v = -\cos x$

$= -\sin x \cdot \cos x + \int \underbrace{\cos^2 x}_{1-\sin^2 x}\, dx = -\sin x\cdot\cos x + x - \int \sin^2 x\, dx$

$\boxed{\Rightarrow \int \sin^2 x\, dx = \frac{1}{2}\left(-\sin x\cdot\cos x + x\right) + C}$

III.3.2 Changement de Variable (CdV)

$[F(\varphi(x))]' = F'(\varphi(x))\cdot\varphi'(x) \quad (*) \quad = f(\varphi(x))\cdot\varphi'(x)$ si $F' = f$

Thm : (CdV) : Soit $f \in C^0(I)$, $\varphi \in C^1([a,b])$ t.q. $\varphi([a,b]) \subset I$, alors

1) $\boxed{\int_a^b f(\varphi(x))\cdot\varphi'(x)\, dx = \int_{\varphi(a)}^{\varphi(b)} f(x)\, dx}$

2) et si φ est bijective :

$\boxed{\int_A^B f(x)\, dx = \int_{\varphi^{-1}(A)}^{\varphi^{-1}(B)} f(\varphi(x))\cdot\varphi'(x)\, dx}$

Preuve : 1) $\int_a^b f(\varphi(x))\cdot\varphi'(x)\, dx =$

$\stackrel{(*)}{=} \int_a^b [F(\varphi(x))]'\, dx = [F(\varphi(x))]_{x=a}^{b} = F(\varphi(b)) - F(\varphi(a)) = [F(u)]_{u=\varphi(a)}^{\varphi(b)} = \int_{\varphi(a)}^{\varphi(b)} f(x)\, dx$ □.

en utilisant l'éq. (*)

2) $A = \varphi(a) \iff a = \varphi^{-1}(A)$, $B = \varphi(b) \iff b = \varphi^{-1}(B)$ □.

Exple : $\boxed{\int_\alpha^\beta \cos x\cdot e^{\sin x}\, dx} = \int_{\sin(\alpha)}^{\sin(\beta)} e^x\, dx = [e^x]_{\sin\alpha}^{\sin\beta} = \boxed{e^{\sin\beta} - e^{\sin\alpha}}$

de 1) où φ n'est pas une bijection

$f(x) = \exp(x) \equiv e^x$

$\varphi(x) = \sin(x)$

$e^{\sin x}\cdot\cos x = \exp(\varphi(x))\cdot\varphi'(x)$

Notation: /des étapes par 2):

$$\int_A^B f(u)\,du = \int_{\varphi^{-1}(A)}^{\varphi^{-1}(B)} f(\varphi(x))\,\varphi'(x)\,dx \equiv \int_A^B f(u)\,du$$

x, $\varphi(x) = u \Rightarrow f(u)$, $\varphi'(x)\,dx \Rightarrow du$

étape
1) $u = \varphi(x)$ nouvelle variable $\Updownarrow$ $x = \varphi^{-1}(u)$

2) $du \overset{\text{Déf.}}{=} \varphi'(x)dx \iff dx = (\varphi^{-1}(u))'\,du = \dfrac{1}{\varphi'(\varphi^{-1}(u))}\,du$

3) Bornes

x	u
$\varphi^{-1}(B)$	B
$\varphi^{-1}(A)$	A

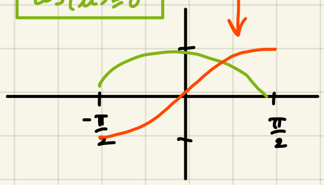
CdV 2) **Exple:** $I = \int_{-1}^{1} \sqrt{1-x^2}\,dx$ ($\sqrt{1-x^2} \to \cos(u)$, $dx \to \cos(u)\,du$)

* il faut changer aussi x et bornes !

⚠ $\sqrt{y^2} = |y|$ module

étape 1 : $x = \sin(u)$ (Nouvelle variable) $\Longrightarrow 1 - x^2 = 1 - \sin^2 u = \cos^2(u)$

$f(x) = \sqrt{1-x^2} = \sqrt{\cos^2(u)} = |\cos(u)|$
$= \cos(u)$ pour $u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$



$[-\frac{\pi}{2}, \frac{\pi}{2}] \xrightarrow{\sim} [-1, 1]$ une bijection

étape 2 : $dx = d(\sin(u)) = \sin'(u)\cdot du = \cos(u)\,du$

étape 3:

x	u
1	$\pi/2$
-1	$-\pi/2$

$$I = \int_{-1}^{1} \sqrt{1-x^2}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\cos^2(u)}_{\frac{1}{2}}\,du = \frac{\pi}{2} \quad (\cos^2 + \sin^2)$$

Rq: CdV $x \leftrightarrow u$ sans bijection pour trouver des primitives, possible mais avec prudence ⚠

Exple: ci-dessous

Rq: $J := \int_{\alpha}^{\alpha + k\cdot\frac{\pi}{2}} \cos^2 x\,dx = \int_{\alpha}^{\alpha+k\frac{\pi}{2}} \underbrace{\sin^2 x}_{1-\cos^2 x}\,dx \Rightarrow 2J = \int_0^{\alpha+k\frac{\pi}{2}} 1\,dx = k\,\frac{\pi}{2} \Rightarrow J = \dfrac{k\frac{\pi}{2}}{2}$ ($k \in \mathbb{Z}^*$)

OU $\int_0^{\pi} \cos^2 x\,dx = \int_0^{\pi} \left(\frac{\cos^2 x}{2} + \frac{\cos^2 x}{2}\right) dx = \int_0^{\pi} \frac{\cos^2 x + \sin^2 x}{2}\,dx = \frac{\pi}{2}$

Exple: (Rq) $F(x) = \int e^{\sin x}\,\cos(x)\,dx = \int e^u\,du = e^u + C \overset{!}{=} e^{\sin x} + C$

$u = \sin x$
$du = \cos x\,dx$

III.3.3 Chercher primitive de la forme ...

Rq: D'intégrer est un peu un art (pas de procédure fixé) et souvent $\exists$ plusieurs méthodes pour y arriver (dont IPP, CdV).

Exple:
1) $\int x^2 e^{2x}\,dx$ $\quad x^2 e^{2x} \to \tilde{P}(x)\exp(\alpha x)$ avec $\deg \tilde{P} = 2$, $\alpha = 2$

(1) Primitives de $\tilde{P}(x)e^{\alpha x}$ est de la forme $P(x)e^{\alpha x}$ avec $\deg P = \deg \tilde{P}$

dans l'exple:
$P(x)e^{2x} = (ax^2 + bx + c)e^{2x}$

$\left[(ax^2 + bx + c)e^{2x}\right]' = (2ax^2 + 2(a+b) + 2c + b)e^{2x} \overset{!}{=} x^2 e^{2x}$ ($ax^2 \to 2ax$)

alors $\iff 2aX^2 + 2(a+b)X + 2c + b \overset{!}{=} X^2 + 0\cdot X + 0\cdot 1$

$\iff \begin{cases} 2a = 1 \\ a + b = 0 \\ 2c + b = 0 \end{cases}$ $\quad a = \frac{1}{2}$, $b = -\frac{1}{2}$, $c = \frac{1}{4}$

$\Rightarrow \int x^2 e^{2x}\,dx = \left(\frac{1}{2}X^2 - \frac{1}{2}X + \frac{1}{4}\right)e^{2x} + \underline{C}$

"à mon avis, un peu plus rapide que IPP ici"

2) $\int \sin(2x) e^{3x} dx$ $\quad (e^{3x})' = 3 \cdot e^{3x}$

la méthode $\rightsquigarrow$ $\left(\left[A \cdot \sin(2x) + B\cos(2x)\right] e^{3x}\right)' = \left[(3A - 2B)\sin(2x) + (3B + 2A)\cos(2x)\right] e^{3x} \overset{!}{=} \sin(2x) e^{3x}$

$(\sin(2x))' = 2\cos(2x)$, $\quad (\cos(2x))' = -\sin \cdot 2x$, $\quad -2B$

$\iff 3A - 2B = 1$ et $2A + 3B = 0 \iff \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \iff \begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

(2) Pour $\boxed{\sin(\beta x) \cdot e^{\alpha x}}$ et/ou $\cos(\beta x) \cdot e^{\alpha x}$
on cherche une solution de la forme:
$A \cdot \sin(\beta x) \cdot e^{\alpha x} + B \cdot \cos(\beta x) \cdot e^{\alpha x}$

III.4 Intégration de fractions rationelles

$\int \frac{P(x)}{Q(x)} dx$ $\quad$ 1er étape: division euclidienne et D.E.S. du reste.

résultat:

$$\underbrace{\int \tilde{P}(x)\, dx}_{1} + \underbrace{\int \frac{A}{x - c_1} dx + \int \frac{\tilde{A}}{(x - c_1)^2} dx}_{2} + \dots + \underbrace{\int \frac{B + Cx}{x^2 + px + q} dx + \int \frac{\tilde{B} + \tilde{C}x}{(x^2 + px + q)^2} dx}_{\Delta \equiv p^2 - 4q < 0 \quad 3}$$

2me étape:

Pour 1 et 2:

$$\int x^\alpha dx = \begin{cases} \ln(|x|) + C, & \text{si } \alpha = -1 \\ \frac{x^{\alpha+1}}{\alpha+1} + C, & \text{si } \alpha \neq -1 \end{cases}$$

(où définie)

$$\Longrightarrow 1 \quad \int \underbrace{(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n)}_{\sum_{k=0}^{n} a_k \cdot x^k} dx = \sum_{k=0}^{n} a_k \cdot \int x^k \cdot dx = \boxed{\left(\sum_{k=0}^{n} \frac{a_k}{k+1} \cdot x^{k+1}\right) + C}$$

$\underset{n>1}{\Longleftarrow}$ $(n \geq 2)$

$$\boxed{\int \frac{1}{x^n} dx} = \int x^{-n} dx = \frac{x^{-n+1}}{-n+1} + C = \frac{1}{1-n} \cdot \frac{1}{x^{n-1}} + C \equiv \boxed{-\frac{1}{n-1} \cdot \frac{1}{x^{n-1}} + C}$$

$\Longrightarrow$ 2 i) $\int \frac{1}{x - c} dx = \ln(|x - c|) + C$

(par exple avec CdV $u = x - c$, $du = dx$)

ii) $\int \frac{1}{(x-c)^n} dx = \frac{-1}{n-1} \frac{1}{(x-c)^{n-1}} + C$ $\quad n \geq 2$

$\longrightarrow \int \frac{1}{x^n} dx = -\frac{1}{n-1} \cdot \frac{1}{x^{n-1}} + C$

$\hookrightarrow n = 2: \int \frac{1}{x^2} dx = -\frac{1}{x} + C$

$n = 3: \int \frac{1}{x^3} dx = -\frac{1}{2} \cdot \frac{1}{x^2} + C$

pour 3 il faut TRAVAILLER!

Stratégie: i) $\int \frac{1}{x^2 + px + q} dx \xrightarrow{\text{CdV}} \int \frac{1}{1 + x^2} dx$

Pour avoir de à,

ii) $\int \frac{x}{(x^2 + px + q)^n} dx = \frac{1}{2} \int \frac{2x + p}{(x^2 + px + q)^n} dx - \frac{p}{2} \int \frac{1}{(x^2 + px + q)^n} dx$

$n \in \mathbb{N}^*$

Rq: $(x^2 + px + q)' = 2x + p$

$\oplus$ $\boxed{\int \frac{2x + p}{(x^2 + px + q)^n} dx = \int \frac{\varphi'(x)}{\varphi(x)^n} dx \overset{\text{CdV}}{=} -\frac{1}{n-1} \cdot \frac{1}{\varphi(x)^{n-1}} + C}$

$f(x) = \frac{1}{x^n}$, $\varphi(x) = x^2 + px + q$

$\ominus$ $\int \frac{1}{(x^2 + px + q)^n} dx \begin{cases} \text{pour } n = 1 \rightarrow \text{i)} \\ n > 1 \text{ plus tard.} \end{cases}$

Exple: $I = \displaystyle\int_0^2 \frac{3}{x^3+1}\,dx$ — Type exam

étape 1 : décomposer $\frac{3}{x^3+1}$ en éléments simples.

$$x^3+1 = (x+1)\underbrace{(x^2-x+1)}_{\Delta = 1-4 = -3}$$

Substitution :

$$\frac{3}{x^3+1} = \frac{A}{x+1} + \frac{B+Cx}{x^2-x+1} = \frac{1}{x+1} + \frac{2-x}{x^2-x+1}$$

$$\boxed{A} = \left(\underbrace{\frac{3}{x^2-x+1}}_{1+1+1=3}\right)\Bigg|_{x=-1} = \boxed{1}$$

$x=0$: $3 = A+B$ $\qquad \boxed{B=2}$

$x=1$: $\frac{3}{2} = \frac{1}{2} + \frac{2+C}{1}$

$\Rightarrow \boxed{C=-1}$

$$I = \underbrace{\Big[\ln(|x+1|)\Big]_0^2}_{\ln(3)} - \frac{1}{2}\int_0^2 \frac{2x}{x^2-x+1}\,dx + 2\int_0^2 \frac{1}{x^2-x+1}\,dx = \ln(3) - \frac{1}{2}\int_0^2 \frac{2x-1}{x^2-x+1} + \frac{3}{2}\int_0^2 \frac{1}{x^2-x+1}\,dx$$

$\ln(2+1) - \cancel{\ln(0)}$

$(x^2-x+1)' = 2x-1$

$-1 \cdot -\frac{1}{2} = +\frac{1}{2}$

$\frac{2}{1}\ \frac{4}{2}$ $-\frac{1}{2}$ $\frac{3}{2}$

$$\int \frac{2x-1}{x^2-x+1}\,dx = \ln\underbrace{(x^2-x+1)}_{>0} + C$$

⚠ $u = x^2-x+1$ pour $x \in [0,2]$ n'est pas une bijection $\longrightarrow$ CdV 1)

$$I = \underbrace{\ln(3) - \frac{1}{2}\Big[\ln(\underbrace{x^2-x+1}_{4-2+1=3})\Big]_0^2}_{\ln(3) - \frac{1}{2}(\ln(3))} + \frac{3}{2} \quad := J$$

$\frac{1}{2}\ln(3) = \ln(\sqrt{3})$

où $J := \displaystyle\int_0^2 \frac{1}{x^2-x+1}\,dx$ $\qquad \boxed{I = \ln(\sqrt{3}) + \frac{3}{2}\cdot\frac{\sqrt{3}}{2}\cdot J}$

$$J = \int_0^2 \frac{1}{x^2-x+1}\,dx = \int_{-1/2}^{3/2} \frac{1}{u^2+\frac{3}{4}}\,du = \frac{4}{3}\,\frac{\sqrt{3}}{2}\int_{-1/\sqrt{3}}^{\sqrt{3}} \frac{1}{v^2+1}\,dv$$

But effacer-le

$$x^2-x+1 = \underbrace{\left(x-\frac{1}{2}\right)^2}_{x^2-x+\frac{1}{4}} + \frac{3}{4}$$

CdV

① $u := x - \frac{1}{2}$ (une bijection !)

② $du = dx$

③

x	u
2	3/2
0	-1/2

Bornes

$u^2 + \frac{3}{4} = \frac{3}{4}\Big(\underbrace{\frac{4}{3}u^2}_{v^2}+1\Big)$

où $v = \frac{2}{\sqrt{3}}u$, $dv = \frac{2}{\sqrt{3}}du$

u	v
3/2	$\sqrt{3}$
-1/2	$-1/\sqrt{3}$

$$J = \frac{4}{3}\Big[\arctan(v)\Big]_{v=-\frac{1}{\sqrt{3}}}^{\sqrt{3}} = \frac{4}{3}\Big(\underbrace{\arctan(\sqrt{3})}_{\pi/3} + \underbrace{\arctan\left(\tfrac{1}{\sqrt{3}}\right)}_{\pi/6}\Big) = \frac{2\pi}{3}$$

$-\arctan\left(-\frac{1}{\sqrt{3}}\right) = +\arctan\left(\frac{1}{\sqrt{3}}\right)$ car arctan est fonct° impaire.

alors $J = \frac{2\pi}{3}$

et $\boxed{I = \ln(\sqrt{3}) + \frac{\sqrt{3}}{2}\pi}$