

Feuille 1 : Calculs de base Éléments de correction – Parties B et C

B. Exercices supplémentaires (plus difficiles)

Exercice 6 – Identité avec des radicaux

Soient $x \geq 0$ et $y \geq 0$ deux réels, alors on a

$$\begin{aligned} \sqrt{x + x^{\frac{2}{3}}y^{\frac{1}{3}}} + \sqrt{y + x^{\frac{1}{3}}y^{\frac{2}{3}}} &= \sqrt{\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)x^{\frac{2}{3}}} + \sqrt{\left(y^{\frac{1}{3}} + x^{\frac{1}{3}}\right)y^{\frac{2}{3}}} \\ &= \sqrt{x^{\frac{1}{3}} + y^{\frac{1}{3}}} \left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right) \\ &= \left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^{\frac{3}{2}} \\ &= \sqrt{x^{\frac{1}{3}} + y^{\frac{1}{3}}}^3. \end{aligned}$$

Exercice 7 – Histoire de carrés

Soient a, b, c et d des réels tels que $a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da$, alors on a :

$$\begin{aligned} (a-b)^2 + (b-c)^2 + (c-d)^2 + (d-a)^2 &= a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2cd + d^2 + d^2 - 2da + a^2 \\ &= 2(a^2 + b^2 + c^2 + d^2) - 2(ab + bc + cd + da) \\ &= 0, \end{aligned}$$

car $a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da$.

$$\text{Or, } (a-b)^2 + (b-c)^2 + (c-d)^2 + (d-a)^2 = 0 \Leftrightarrow \begin{cases} a-b=0 \\ b-c=0 \\ c-d=0 \\ d-a=0 \end{cases} \Leftrightarrow a=b=c=d$$

Exercice 8 – Identité de Gauss

Soient a, b et c trois réels, alors on a

$$\begin{aligned} (a+b+c)^3 &= a^3 + b^3 + c^3 + 6abc + 3(ab^2 + ac^2 + a^2b + a^2c + cb^2 + bc^2) \\ &= a^3 + b^3 + c^3 + 3(a^2(b+c) + b^2(a+c) + c^2(a+b)) + 6abc \end{aligned}$$

Si $a+b+c=0$, alors $(a+b+c)^3=0$ et $b+c=-a$, $a+c=-b$, $a+b=-c$.

Donc $0 = a^3 + b^3 + c^3 + 3(-a^3 - b^3 - c^3) + 6abc \Leftrightarrow a^3 + b^3 + c^3 = 3abc$.

C. Exercices d'entraînement (à faire en autonomie)

Exercice 9 – Fractions

1. $-\frac{1}{2} + \frac{7}{4} + \frac{5}{12} = -\frac{6}{12} + \frac{21}{12} + \frac{5}{12} = \frac{20}{12} = \frac{5}{3}$.
2. $-3 + \frac{4}{5} + \frac{2}{3} = -\frac{45}{15} + \frac{12}{15} + \frac{10}{15} = -\frac{23}{15}$.
3. $\frac{2}{4} - \frac{3}{\frac{5}{7}} + \frac{1}{30} = \frac{2}{12} - \frac{21}{5} + \frac{1}{30} = \frac{1}{6} - \frac{21}{5} + \frac{1}{30} = \frac{5}{30} - \frac{126}{30} + \frac{1}{30} = -\frac{120}{30} = -4$.
4. $\frac{54}{157} \left[\left(\frac{2}{3}\right)^3 - 2 - \left(-\frac{1}{2}\right)^2 \right] = \frac{54}{157} \left[\frac{8}{27} - 2 - \frac{1}{4} \right] = \frac{54}{157} \left[\frac{32}{108} - \frac{216}{108} + \frac{27}{108} \right] = -\frac{54}{157} \cdot \frac{157}{108} = -\frac{54}{108} = -\frac{1}{2}$.

$$5. \frac{1}{\frac{4}{5}} \cdot \frac{\frac{2}{5}}{\frac{1}{3}} \cdot \frac{\left(\frac{2-\frac{1}{8}}{\frac{1}{3}-1}\right)^2}{\left(\frac{1}{3}-1\right)^3} + \frac{123}{512} = \frac{5}{4} \cdot \frac{4}{5} \cdot \frac{\left(\frac{17}{8}\right)^2}{\left(-\frac{2}{3}\right)^3} + \frac{123}{512} = \frac{17^2}{8^2} \cdot \frac{3^3}{(-2)^3} + \frac{123}{512} = \frac{17^2 \cdot 27}{-8^3} + \frac{123}{512} = -\frac{289 \cdot 27}{512} + \frac{123}{512} = -\frac{7803}{512} + \frac{123}{512} = -\frac{7680}{512} = -15.$$

Exercice 10 – Puissances

1. $\frac{27^{-1} \cdot 4^2}{3^{-4} \cdot 2^4} = \frac{(3^3)^{-1} \cdot (2^2)^2}{3^{-4} \cdot 2^4} = \frac{3^{-3} \cdot 2^4}{3^{-4} \cdot 2^4} = \frac{3^{-3}}{3^{-4}} = \frac{3^4}{3^3} = 3.$
2. $\left((-2)^3 + (-5)^2 - 1\right)^{-\frac{1}{2}} = (-8 + 25 - 1)^{-\frac{1}{2}} = 16^{-\frac{1}{2}} = \frac{1}{\sqrt{16}} = \frac{1}{4}.$
3. $\frac{(10^5 \cdot 10^{-3})^5}{(10^{-5} \cdot 10^3)^{-3}} = \frac{(10^2)^5}{(10^{-2})^{-3}} = \frac{10^{10}}{10^6} = 10^4 = 10000.$
4. $\frac{3^{22} + 3^{21}}{3^{22} - 3^{21}} = \frac{3^{21}(3+1)}{3^{21}(3-1)} = \frac{4}{2} = 2.$
5. $\frac{55^2 \cdot 121^{-2} \cdot 125^2}{275 \cdot 605^{-2} \cdot 25^4} = \frac{5^2 \cdot 11^2 \cdot (11^2)^{-2} \cdot (5^3)^2}{5^2 \cdot 11 \cdot (5 \cdot 11^2)^{-2} \cdot (5^2)^4} = \frac{5^8 \cdot 11^{-2}}{5^8 \cdot 11^{-3}} = 11.$

Exercice 11 – Racines carrées

1. $\sqrt{(-5)^2} - \sqrt{5^2} = \sqrt{25} - 5 = 5 - 5 = 0.$
2. $\sqrt{(3-\pi)^2} = |3-\pi| = \pi - 3$ car $\pi > 3.$
3. $(3+\sqrt{7})^2 - (3-\sqrt{7})^2 = (3+\sqrt{7}+3-\sqrt{7})(3+\sqrt{7}-3+\sqrt{7}) = 6 \cdot 2\sqrt{7} = 12\sqrt{7}.$
4. $\frac{2+\sqrt{3}}{1-\sqrt{3}} = \frac{(2+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})} = \frac{2+2\sqrt{3}+\sqrt{3}+3}{1-(\sqrt{3})^2} = \frac{5+3\sqrt{3}}{-2} = -\frac{5}{2} - \frac{3}{2}\sqrt{3}.$
5. $\frac{5+2\sqrt{6}}{\sqrt{2}+\sqrt{3}} + \frac{5-2\sqrt{6}}{\sqrt{2}-\sqrt{3}} = \frac{(5+2\sqrt{6})(\sqrt{2}-\sqrt{3})+(5-2\sqrt{6})(\sqrt{2}+\sqrt{3})}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})} = \frac{10\sqrt{2}-4\sqrt{6}\sqrt{3}}{2-3} = -10\sqrt{2}+4\sqrt{18} = -10\sqrt{2}+12\sqrt{2} = 2\sqrt{2} = \sqrt{8}.$

Exercice 12 – Identités remarquables

Soient x, y, z et t quatre réels tels que les expressions suivantes soient bien définies.

1. $\frac{x^4-1}{x^2+1} + 1 = \frac{(x^2+1)(x^2-1)}{x^2+1} + 1 = x^2 - 1 + 1 = x^2.$
2. $(xt+yz)^2 + (xz-yt)^2 - (x^2+y^2)(z^2+t^2) = x^2t^2 + y^2z^2 - 2xtyz + x^2z^2 + y^2t^2 - 2xzyt - x^2z^2 - x^2t^2 - y^2z^2 - y^2t^2 = 0.$
3. $\frac{x^3-1}{x-1} - (x-1)^2 = \frac{(x-1)(x^2+x+1)}{x-1} - (x-1)^2 = x^2 + x + 1 - x^2 + 2x - 1 = 3x.$
4. $\frac{2x+3}{\frac{1}{x^2-4} - \frac{2}{x-2}} = \frac{2x+3}{\frac{1}{x^2-4} - \frac{2(x+2)}{x^2-4}} = \frac{2x+3}{\frac{-2x-3}{x^2-4}} = -x^2 + 4.$
5. $\frac{(x+y)^4 - (x-y)^4}{4xy} - 2(x-y)^2 = \frac{x^4+4x^3y+6x^2y^2+4xy^3+y^4 - x^4+4x^3y-6x^2y^2+4xy^3-y^4}{4xy} - 2(x-y)^2 = \frac{8x^3y+8xy^3}{4xy} - 2(x-y)^2 = \frac{8x^3y+8xy^3}{4xy} - 2(x^2-2xy+y^2) = 4xy.$