Q. du CM :

a) $\mathcal{F}$ est libre et génératrice

libre : $\sum_{i=1}^{n} \lambda_i \cdot u_i = 0 \Rightarrow (\lambda_1, \dots, \lambda_n) = (0, \dots, 0) \in \mathbb{K}^n$

génératrice : $\forall x \in E \quad \exists (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n$ t.q. $x = \sum_{i=1}^{n} \lambda_i u_i$.

b) F, G s.e.v. de E.

$$\dim(F+G) = \dim(F) + \dim(G) - \dim(F \cap G)$$

Ex. 1 :

a) OUI, b) NON, c) OUI, d) NON, e) NON, f) NON

Rq. : a) $= \mathbb{R}^3$, c) $= \{0\} \subset \mathbb{R}^3$, d) $= \phi$

$\mathbb{R}^3 \setminus \{0\}$ ou l'intérieur ou l'extérieur d'une boule ne sont pas des espaces vectoriels (cf. CM7, II.3.1 ?)

Ex. 2

a)

$$\left(\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 2 & 0 \\ 1 & 1 & 10 & -1 & -4 & 0 \\ 2 & 2 & 2 & 4 & 1 & 0 \end{array}\right) \begin{array}{l} \\ \leftarrow -1 \\ \leftarrow -2 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 2 & 0 \\ 0 & 0 & 12 & -4 & -6 & 0 \\ 0 & 0 & 6 & -2 & -3 & 0 \end{array}\right) \begin{array}{l} \\ 1 \cdot \frac{1}{2} \\ \leftarrow -2 \end{array}$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -2 & +1 & 3 & 2 & 0 \\ 0 & 6 & 0 & -2 & -3 & 0 \end{array}\right) (*) \Rightarrow \text{rg } 2$$

$$\begin{array}{ccccc} x_1 & x_3 & x_2 & x_4 & x_5 \\ & & \uparrow & \uparrow & \uparrow \\ & & \lambda_1 & \lambda_2 & \lambda_3 \end{array}$$

$x_2 = \lambda_1, \; x_4 = \lambda_2, \; x_5 = \lambda_3$

$6x_3 = 2\lambda_2 + 3\lambda_3 \Rightarrow x_3 = \frac{1}{3}\lambda_2 + \frac{1}{2}\lambda_3$

$x_1 - 2x_3 + x_2 + 3x_4 + 2x_5 = 0$

$x_1 = 2/3 \cdot \lambda_2 + \lambda_3 - \lambda_1 - 3\lambda_2 - 2\lambda_3 = -\lambda_1 - \frac{7}{3}\lambda_2 - \lambda_3$

$$\Rightarrow \vec{x} \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \lambda_1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -\frac{7}{3} \\ 0 \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ 0 \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \Rightarrow \boxed{\ker A = \text{Vect}\left(\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 0 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} \right)}$$

b) $(*) \Rightarrow \boxed{rg(A) = 2} = \dim(\operatorname{im} A) \Rightarrow \boxed{\operatorname{im}(A) = \operatorname{Vect}\left(\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}3\\-1\\4\end{pmatrix}\right)}$

$\Rightarrow$ 2 vecteurs de colonne de la matrice A non-colinéaires

c) $\begin{pmatrix}1\\1\\2\end{pmatrix} \wedge \begin{pmatrix}3\\-1\\4\end{pmatrix} = \begin{pmatrix}6\\2\\-4\end{pmatrix} = 2 \cdot \begin{pmatrix}3\\1\\-2\end{pmatrix} \Rightarrow \boxed{\operatorname{im}(A) = \{(x,y,z) \in \mathbb{R}^3 \mid 3x + y - 2z = 0\}}$

<u>Ex. 3</u>

a) • Le polynôme $0 \in F$

• $\forall P, Q \in F,\ \forall \lambda \in \mathbb{R} : (P + \lambda Q)(0) \overset{\text{déf}}{=} \underbrace{P(0)}_{0} + \lambda \underbrace{Q(0)}_{0} = 0$ ✓

$(P + \lambda Q)'(1) \overset{\text{déf}}{=} \underbrace{P'(1)}_{0} + \lambda \underbrace{Q'(1)}_{0} = 0$ ✓ #

b) $aX^3 + bX^2 + cX + d = P \in \mathbb{R}^3[X] \Rightarrow P(0) = \boxed{d \overset{!}{=} 0}$

$P'(1) = 3a + 2b + c \overset{!}{=} 0 \Leftrightarrow \boxed{c = -3a - 2b}$

$\Rightarrow P \in F \Leftrightarrow P = aX^3 + bX^2 - (3a + 2b)X = a(X^3 - 3X) + b(X^2 - 2X)$ $\forall a, b \in \mathbb{R}$

$\Rightarrow \boxed{F = \operatorname{Vect}(X^3 - 3X,\ X^2 - 2X)}$

c) $\dim F = 2$, $\dim G = 2$ $\quad G = \operatorname{Vect}(X^3 - X - 1,\ X^2 + 1)$

$\boxed{F \cap G = \{0\}}$ car $\alpha(X^3 - 3X) + \beta(X^2 - 2X) = \gamma(X^3 - X - 1) + \delta(X^2 + 1)$

(**) $\Leftrightarrow X^3 \cdot (\alpha - \gamma) + X^2 \cdot (\beta - \delta) + X(-3\alpha - 2\beta + \gamma) + 1 \cdot (\gamma - \delta) \overset{!}{=} 0$

$\Leftrightarrow \underbrace{\alpha = \gamma,\ \beta = \delta = \gamma}_{\alpha = \beta = \gamma = \delta}$ et $-3\alpha - 2\beta + \gamma = -4\alpha \overset{!}{=} 0$

$\Rightarrow \alpha = \beta = \gamma = \delta = 0$ ✓

$\Rightarrow \dim(F + G) = 2 + 2 = 4$

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$F + G \subseteq \mathbb{R}_3[X]$, $\dim \mathbb{R}_3[X] = 4$ $\Bigg\} \Rightarrow F + G = \mathbb{R}_3[X]$

(**) $\Rightarrow F + G = F \oplus G \Rightarrow \boxed{\mathbb{R}_3[X] = F \oplus G}$ #

Ex. 4 a) NON, b) OUI, c) OUI, d) NON, e) NON, f) NON

Rqs.: a) $1 \notin E$, b) $\dim E = 2$ $(\underset{\mathbb{R}_2[X]}{3} - \underset{\text{une éq. lin.}}{1})$ et $1+X$ ~~et~~ X^2 ne sont pas colinéaires

c) $\mathrm{Vect}(2X^2, X^2+X+1) = \mathrm{Vect}(X^2, X^2+X+1) = \underbrace{\mathrm{Vect}(X^2, X+1)}_{b)\,!}$

d) $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{vmatrix} = 14 - 14 = 0 \Rightarrow$ NON

e) $\dim E = 3 - 1 = 2$, $|\mathcal{F}| = 3$ ↯

f) $A := \alpha \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \beta \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \delta \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} \alpha+2\beta & * \\ \alpha+2\beta & * \end{pmatrix}$

$\Rightarrow A_{11} = A_{21}$ mais il existe des matrices $A \in \mathcal{M}_2(\mathbb{R})$ qui ne satisfont pas cette restriction $\Rightarrow$

$\Rightarrow \mathcal{F}$ n'est pas génératrice.