

① a) $\forall P, Q \in \mathbb{R}_2[X], \forall \lambda \in \mathbb{R}$:

$$\begin{aligned} f(P+\lambda Q) &= 2X(P+\lambda Q)' + (P+\lambda Q)(1) \cdot X^2 = \\ &= 2X(P' + \lambda Q') + (P + \lambda Q)X^2 = \\ &= 2XP' + P(1)X^2 + \lambda(2XQ' + Q(1)X^2) = \\ &= f(P) + \lambda f(Q) \quad \# \end{aligned}$$

b) $f(1) = X^2, f(X) = 2X + X^2, f(X^2) = 5X^2$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 5 \end{pmatrix} \quad (\Rightarrow \text{rg}(A) = 2)$$

c) ker A: $\begin{array}{ccc|c} 1 & 1 & 5 & 0 \\ 0 & 2 & 0 & 0 \end{array} \Rightarrow \begin{array}{l} z = \lambda \\ y = 0 \\ x = -5\lambda \end{array}$

$$\ker A = \text{Vect} \left(\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$\text{im } A = \text{Vect} \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right) = \text{Vect} \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) = \text{Vect}(e_2, e_3)$$

d) $\ker f = \text{Vect}(-5 + X^2)$
 $\text{im } f = \text{Vect}(X, X^2)$

$$\left(\text{car } \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} \cong -5 \cdot 1 + 1 \cdot X^2 \text{ etc.} \right)$$

$$\left(\text{Rq.: } \ker f + \text{im } f = \ker f \oplus \text{im } f = \mathbb{R}_2[X] \right)$$