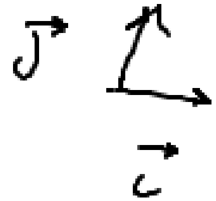
$\boxed{\text{Aires}}$ $\qquad$ $\boxed{K = \mathbb{R}}$

Contexte : géométrie affine dans un plan.

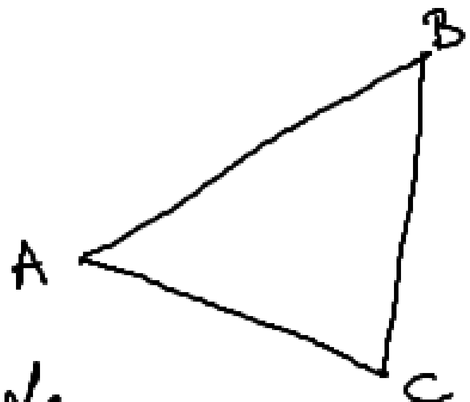
$\mathcal{P}$ espace affine de dimension 2 sur $\mathbb{R}$.

On fixe une base $(\vec{i}, \vec{j})$ de $\vec{\mathcal{P}}$.



Déf

$$\mathcal{A}(ABC) = \frac{1}{2} \det_{(\vec{i}, \vec{j})} (\vec{AB}, \vec{AC})$$



est l'aire algébrique du triangle orienté ABC (l'ordre des sommets est important).

Remarques

* $\mathcal{A}(ACB) = \frac{1}{2} \det_{(\vec{i},\vec{j})} (\vec{AC}, \vec{AB})$

$= -\frac{1}{2} \det_{(\vec{i},\vec{j})} (\vec{AB}, \vec{AC}) = -\mathcal{A}(ABC).$

* $\mathcal{A}(ABC) = \mathcal{A}(BCA) = \mathcal{A}(CAB)$

$\mathcal{A}(ABC) = \frac{1}{2} \det_{(\vec{i},\vec{j})} (\vec{AB}, \vec{AC}) = \frac{1}{2} \det_{(\vec{i},\vec{j})} (-\vec{BA}, \vec{AB}+\vec{BC})$

$= \frac{1}{2} \det_{(\vec{i},\vec{j})} (-\vec{BA}, \vec{AB}) + \frac{1}{2} \det_{(\vec{i},\vec{j})} (-\vec{BA}, \vec{BC})$

$= 0 - \frac{1}{2} \det_{(\vec{i},\vec{j})} (\vec{BA}, \vec{BC})$

$= \frac{1}{2} \det (\vec{BC}, \vec{BA}) = \mathcal{A}(BCA).$

* Changement de base

$(\vec{i}, \vec{j}) \rightsquigarrow (\vec{i}', \vec{j}')$

P = matrice des coordonnées de $(\vec{i}, \vec{j})$ dans $(\vec{i}', \vec{j}')$.

Les aires algébriques sont multipliées par $\det(P)$:

$$\boxed{\frac{1}{2}\det_{(\vec{i}', \vec{j}')}(\vec{AB}, \vec{AC}) = \det(P) \times \frac{1}{2}\det_{(\vec{i}, \vec{j})}(\vec{AB}, \vec{AC})}$$

En effet, $(\vec{AB}, \vec{AC})_{(\vec{i}', \vec{j}')} = P \times (\vec{AB}, \vec{AC})_{(\vec{i}, \vec{j})}$.

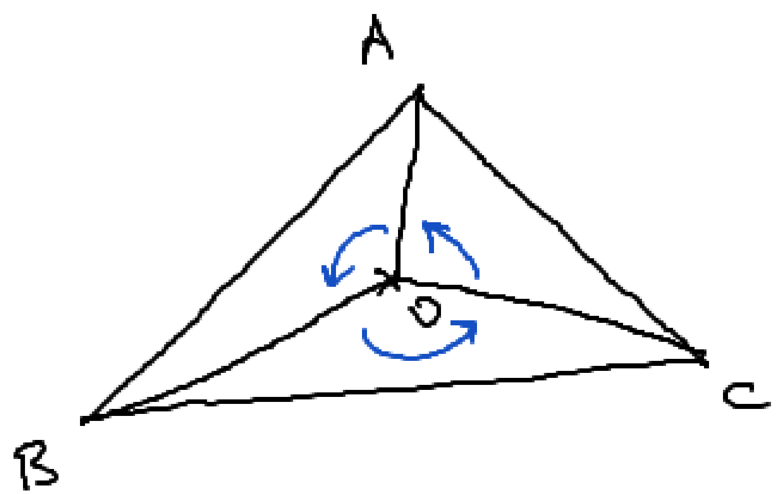
<u>Rq</u> – Les rapports d'aires algébriques ne dépendent pas du choix de la base.

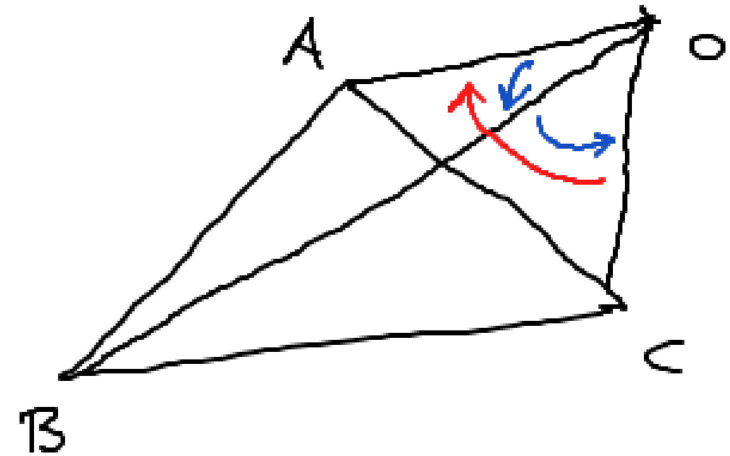
Aire géométrique : $|\mathcal{A}(ABC)| \in \mathbb{R}_{\geq 0}$

Rq. $\mathcal{A}(ABC) = 0 \iff \vec{AB}, \vec{AC}$ colinéaires

$\iff A, B, C$ alignés.

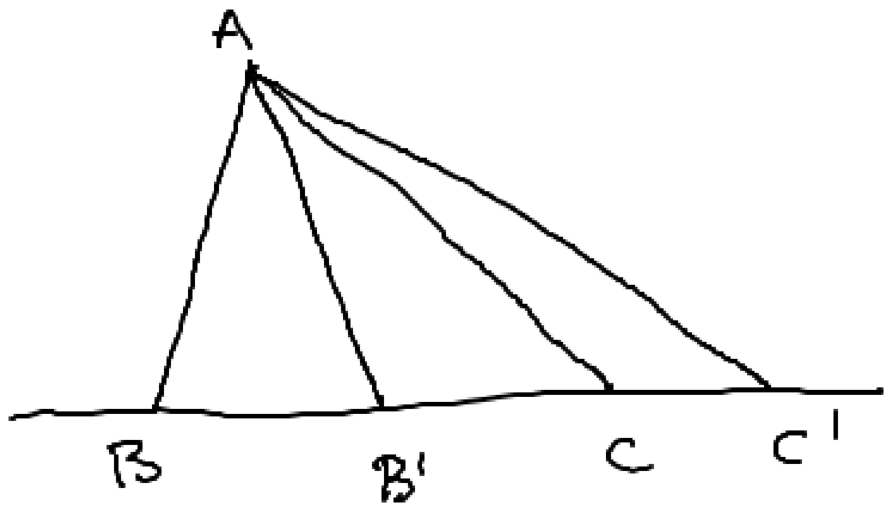
Découpage





$$\mathcal{A}(ABC) = \mathcal{A}(OAB) + \mathcal{A}(OBC) + \mathcal{A}(OCA)$$

Lemme [Proportions]



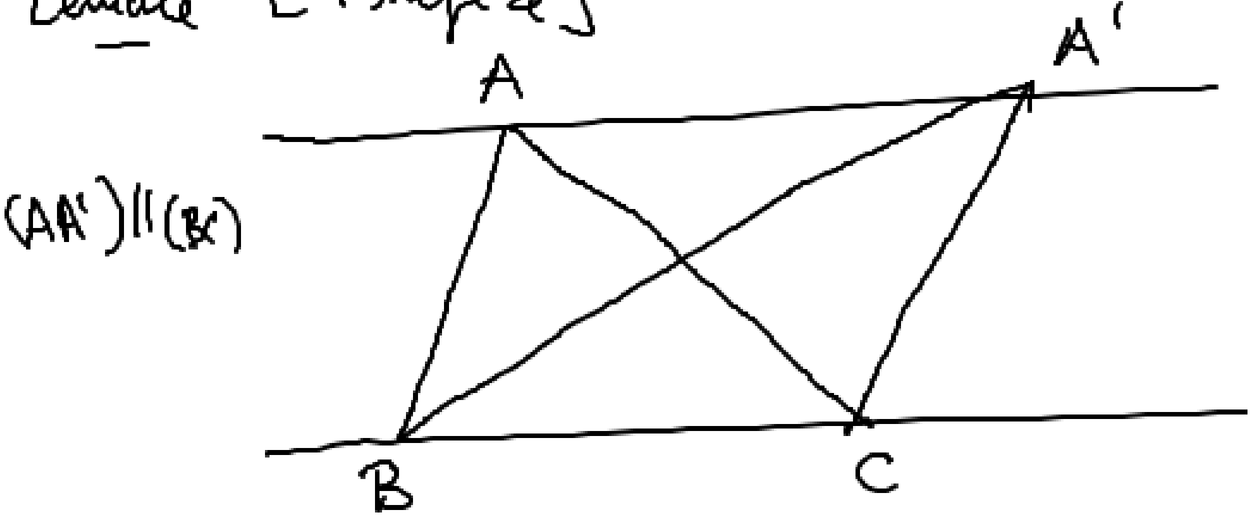
$$\frac{\mathcal{A}(ABC)}{\mathcal{A}(AB'C')} = \frac{\overrightarrow{BC}}{\overrightarrow{B'C'}}$$

Preuve $\quad \overrightarrow{BC} = \lambda \overrightarrow{B'C'}$

$$\begin{aligned}
\mathcal{A}(ABC) = \mathcal{A}(BCA) &= \frac{1}{2}\det(\overrightarrow{BC}, \overrightarrow{BA}) \\
&= \frac{1}{2}\det(\lambda\overrightarrow{B'C'}, \overrightarrow{BB'} + \overrightarrow{B'A}) \\
&= \frac{1}{2}\det(\lambda\overrightarrow{B'C'}, \overrightarrow{B'A}) \\
&= \lambda \cdot \frac{1}{2}\det(\overrightarrow{B'C'}, \overrightarrow{B'A}) = \lambda\, \mathcal{A}(B'C'A) \\
&= \lambda \cdot \mathcal{A}(AB'C')
\end{aligned}$$

$\square$

Lemme [Trapèze]



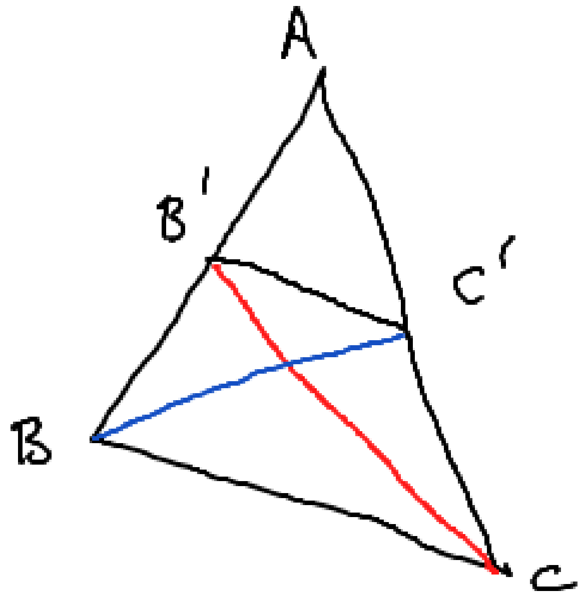
$$\mathcal{A}(ABC) = \mathcal{A}(A'BC)$$

Dém.

$$\begin{aligned}\mathcal{A}(A'BC) &= \mathcal{A}(BCA') \\ &= \frac{1}{2}\det(\overrightarrow{BC}, \overrightarrow{BA'}) \\ &= \frac{1}{2}\det(\overrightarrow{BC}, \overrightarrow{BA} + \overrightarrow{AA'}) \\ &= \frac{1}{2}\det(\overrightarrow{BC}, \overrightarrow{BA}) = \mathcal{A}(BCA) = \mathcal{A}(ABC)\end{aligned}$$

$\square$

Thalès par les aires



$(B'C') \parallel (BC)$

$$\frac{\overrightarrow{AB'}}{\overrightarrow{AB}} \underset{(1)}{=} \frac{\overrightarrow{AC'}}{\overrightarrow{AC}} \underset{(2)}{=} \frac{\overrightarrow{B'C'}}{\overrightarrow{BC}}$$

(1) $$\frac{\overrightarrow{B'B}}{\overrightarrow{AB}} = \frac{\mathcal{A}(CB'B)}{\mathcal{A}(CAB)}$$ [Proportions]

$$= \frac{\mathcal{A}(B'BC)}{\mathcal{A}(CAB)}$$

$$= \frac{\mathcal{A}(C'BC)}{\mathcal{A}(CAB)}$$ [Trapèze]

$$\frac{\overrightarrow{B'B}}{\overrightarrow{AB}} = \frac{\mathcal{A}(BCC')}{\mathcal{A}(BCA)}$$

$$= \frac{\overrightarrow{CC'}}{\overrightarrow{CA}}$$ [Proportions]

$$\boxed{\frac{\overrightarrow{B'B}}{\overrightarrow{AB}} = \frac{\overrightarrow{CC'}}{\overrightarrow{CA}}} \Rightarrow 1 + \frac{\overrightarrow{B'A}}{\overrightarrow{AB}} = 1 + \frac{\overrightarrow{AC'}}{\overrightarrow{CA}}$$

$$\Rightarrow \frac{\overrightarrow{B'A}}{\overrightarrow{AB}} = \frac{\overrightarrow{AC'}}{\overrightarrow{CA}}.$$