Fiche 1

Rappels :

$(E, \langle , \rangle)$ $\qquad \langle , \rangle : E \times E \longrightarrow \mathbb{R}$ produit scalaire

Exemple : $E = \mathbb{R}^n$ $\quad \left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = (x_1\ x_2 \dots x_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^{n} x_i y_i$ produit scalaire standard.

Orthogonalité : $x \perp y$ si $\langle x, y \rangle = 0$.

Norme de x : $\|x\| = \sqrt{\langle x, x \rangle}$

Famille orthogonale : $(e_1, \dots, e_k)$ t.q. $\langle e_i, e_j \rangle = 0$ pour tout $i \neq j$

— " — orthonormée : $\langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{si } i = j \\ 0 & \text{si } i \neq j \end{cases}$

Si $V \subset E$ est une partie de E, $V^\perp = \{ x \in E, \forall v \in V, \langle x, v \rangle = 0 \}$

$V^\perp$ est un sous-espace de E.

Si V est un sous-espace de E, $E = V \oplus V^\perp$

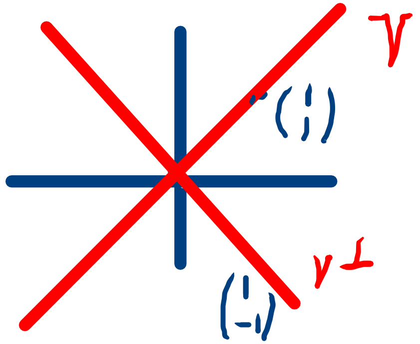
Exemple : $\mathbb{R}^2$, $\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = x_1 y_1 + x_2 y_2 = (x_1 \ x_2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$V = \mathrm{Vect}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$V^\perp = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ; \ x_1 + x_2 = 0 \right\} = \left\{ \begin{pmatrix} t \\ -t \end{pmatrix}, t \in \mathbb{R} \right\}$$

$$= \mathrm{Vect}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\mathbb{R}^2 = V \oplus V^\perp$$



Gram-Schmidt pour 2 vecteurs.

$u, x \in E \setminus \{0\}$ t.q. (u, x) soit libre

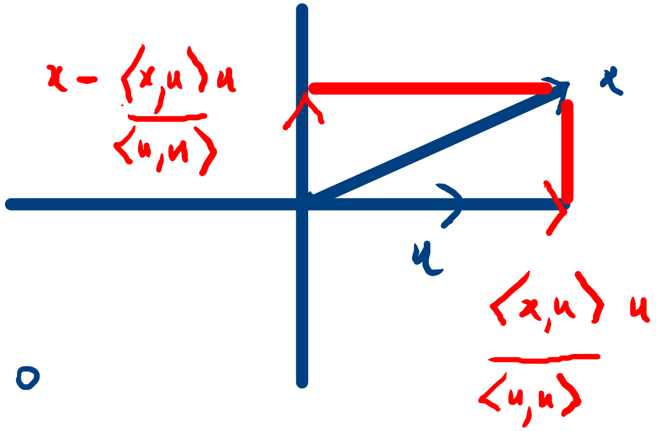
$$E = \mathrm{Vect}(u) \oplus \mathrm{Vect}(u)^\perp$$

$$x = \lambda u + x - \lambda u$$

→ Déterminer $\lambda \in \mathbb{R}$ pour que $\langle x - \lambda u, u \rangle = 0$

$$0 = \langle x, u \rangle - \lambda \langle u, u \rangle \quad \text{i.e.} \quad \lambda = \frac{\langle x, u \rangle}{\langle u, u \rangle}$$

$\left(u, \ x - \dfrac{\langle x, u \rangle}{\langle u, u \rangle} u \right)$ est orthogonale



Exo 1

1.(a) $\mathbb{R}^3$, $\langle , \rangle$ standard. $V = \text{Vect}\begin{pmatrix}2\\-3\\6\end{pmatrix}$

$$V^\perp = \left\{ \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}, \ (x_1\ x_2\ x_3)\begin{pmatrix}2\\-3\\6\end{pmatrix} = 0 \right\}$$

$$= \left\{ \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}, \ 2x_1 - 3x_2 + 6x_3 = 0 \right\}$$

base de $V^\perp = \left(\begin{pmatrix}3\\2\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\end{pmatrix} \right)$

$2x_1 - 3x_2 + 6x_3 = 0$

$2x_1 = 3x_2 - 6x_3$

$\begin{cases} x_3 = 0 \\ x_2 = 2 \\ x_1 = 3 \end{cases}$ $\quad$ $x_2 = 0$, $x_3 = 1$, $x_1 = -3$

Autre base : $\left(\begin{pmatrix}0\\2\\1\end{pmatrix}, \begin{pmatrix}3\\2\\0\end{pmatrix} \right) = (u_1, u_2)$

b o n de $V^\perp$ par Gram-Schmidt : $u_1' = u_1$,

$$u_2' = u_2 - \frac{\langle u_2, u_1' \rangle}{\langle u_1', u_1' \rangle} u_1' = \begin{pmatrix}3\\2\\0\end{pmatrix} - \frac{(3\ 2\ 0)\begin{pmatrix}0\\2\\1\end{pmatrix}}{(0\ 2\ 1)\begin{pmatrix}0\\2\\1\end{pmatrix}} \begin{pmatrix}0\\2\\1\end{pmatrix}$$

$$= \begin{pmatrix}3\\2\\0\end{pmatrix} - \frac{4}{5}\begin{pmatrix}0\\2\\1\end{pmatrix} = \begin{pmatrix}3\\2/5\\-4/5\end{pmatrix}$$

(u_1', u_2') est orthogonale

$v_1 = \dfrac{u_1'}{\|u_1'\|}$, $v_2 = \dfrac{u_2'}{\|u_2'\|}$ $\quad$ (v_1, v_2) est une bon

$\|u_1'\|^2 = \langle u_1', u_1' \rangle = (0\ 2\ 1)\begin{pmatrix}0\\2\\1\end{pmatrix} = 5$ $\quad$ $\|u_1'\| = \sqrt{5}$

$\|u_2'\|^2 = \langle u_2', u_2' \rangle = \frac{1}{25}(15\ 2\ -4)\begin{pmatrix}15\\2\\-4\end{pmatrix} = \frac{49}{5}$ $\quad$ $\|u_2'\| = \frac{7}{\sqrt{5}}$

$\mathbb{R}^3$, $\langle , \rangle$ standard

Exo 1 1.(b) $u_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(u_1, u_2, u_3) base : $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0$
$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$ | pour vous.

1.(c) Construire une bon (e_1, e_2, e_3) à partir de (u_1, u_2, u_3) par Gram-Schmidt.

G-S : $u_1' = u_1$, $u_2' = u_2 - \frac{\langle u_2, u_1' \rangle}{\langle u_1', u_1' \rangle} u_1'$,

$$u_3' = u_3 - \frac{\langle u_3, u_1' \rangle}{\langle u_1', u_1' \rangle} u_1' - \frac{\langle u_3, u_2' \rangle}{\langle u_2', u_2' \rangle} u_2'$$

(u_1', u_2', u_3') orthogonale

$\left(\frac{u_1'}{\|u_1'\|}, \frac{u_2'}{\|u_2'\|}, \frac{u_3'}{\|u_3'\|} \right)$ bon.

$$u_1' = u_1 = \begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$u_2' = \begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{(1\ 0\ 1)\begin{pmatrix}1\\2\\2\end{pmatrix}}{(1\ 2\ 2)\begin{pmatrix}1\\2\\2\end{pmatrix}} \begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{3}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$= \begin{pmatrix}1-\frac{1}{3}\\0-\frac{2}{3}\\1-\frac{2}{3}\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$$u_3' = \begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{(1\ 1\ 0)\begin{pmatrix}1\\2\\2\end{pmatrix}}{9}\begin{pmatrix}1\\2\\2\end{pmatrix} - \cancel{\frac{(1\ 1\ 0)\frac{1}{3}\begin{pmatrix}2\\-2\\1\end{pmatrix}}{\frac{1}{3}(2\ -2\ 1)\ \frac{1}{3}\begin{pmatrix}2\\-2\\1\end{pmatrix}}}^{\,0}\ \frac{1}{3}\begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{3}{9}\begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}\frac{1}{3}\\ \frac{2}{3}\\ \frac{2}{3}\end{pmatrix} = \begin{pmatrix}\frac{2}{3}\\ \frac{1}{3}\\ -\frac{2}{3}\end{pmatrix}$$

$$u_1' = \begin{pmatrix}1\\2\\2\end{pmatrix},\quad u_2' = \frac{1}{3}\begin{pmatrix}2\\-2\\1\end{pmatrix},\quad u_3' = \frac{1}{3}\begin{pmatrix}2\\1\\-2\end{pmatrix}$$

$$\|u_1'\|^2 = 9, \qquad \|u_2'\|^2 = 1 = \|u_3'\|^2$$

BON : $\left(\frac{u_1'}{3},\ u_2',\ u_3'\right)$

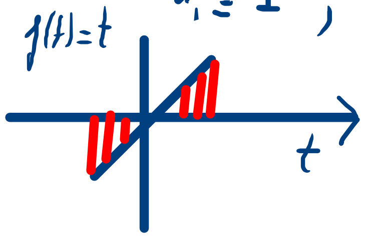
Exo 1 **2.** $E = \mathbb{R}_3[X]$ l'espace des polynômes à coeff réels de degré au plus 3.

$\dim E = 4$, $E = \mathrm{Vect}(1, X, X^2, X^3)$

(a) $$b : E \times E \to \mathbb{R}$$
$$(P, Q) \longmapsto b(P,Q) = \int_{-2}^{2} P(t)\,Q(t)\,dt$$
est un produit scalaire. **Fait déjà**

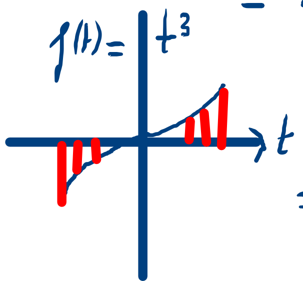
(b) $(u_1 = 1,\ u_2 = X,\ u_3 = X^2,\ u_4 = X^3)$

$\rightsquigarrow$ par G-S construire une bon (v_1, v_2, v_3, v_4).



$$u_1' = 1, \quad u_2' = u_2 - \frac{b(u_2, u_1')}{b(u_1', u_1')}\, u_1' = X - \frac{\overset{0}{\cancel{\int_{-2}^{2} t\,dt}}}{\int_{-2}^{2} dt}\; 1$$
$$= X.$$

$$u_3' = u_3 - \frac{b(u_3, u_1')}{b(u_1', u_1')}\, u_1' - \frac{b(u_3, u_2')}{b(u_2', u_2')}\, u_2'$$



$$= X^2 - \frac{\int_{-2}^{2} t^2\,dt}{\int_{-2}^{2} dt} \cdot 1 - \frac{\overset{0}{\cancel{\int_{-2}^{2} t^3\,dt}}}{\int_{-2}^{2} t^2\,dt}\, X = X^2 - \frac{\left.\frac{t^3}{3}\right|_{-2}^{2}}{4} \cdot 1.$$

$$= X^2 - \frac{1}{3}\,\frac{2^4}{4} \cdot 1 = X^2 - \frac{4}{3}.$$

$$u_4' = x^3 - \frac{\cancel{\int_{-2}^{2} t^3\,dt}}{\int_{-2}^{2} dt}^{0} \cdot 1 - \frac{\int_{-2}^{2} t^4\,dt}{\int_{-2}^{2} t^2\,dt}\, X$$

$$- \frac{\cancel{\int_{-2}^{2} t^3\left(t^2-\frac{4}{3}\right)dt}^{0}}{\int_{-2}^{2}\left(t^2-\frac{4}{3}\right)^2 dt} \cdot \left(X^2 - \frac{4}{3}\right)$$

PAUSE
15h10
→ 15h21

$$\boxed{\begin{array}{l} u_1' = 1 \ , \ u_2' = X \\ u_3' = X^2 - \frac{4}{3} \\ u_4' = X^3 - \frac{12}{5} X \end{array}}$$

$$= X^3 - \frac{\left.\frac{t^5}{5}\right|_{-2}^{2}}{\left.\frac{t^3}{3}\right|_{-2}^{2}} X = X^3 - \frac{2^6}{5}\,\frac{1}{\frac{2^4}{3}}\, X$$

$$= X^3 - \frac{12}{5} X$$

bon

$$v_i = \frac{u_i'}{\sqrt{b(u_i', u_i')}}$$

← ce calcul n'est pas instructif.

Exo 2.

$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $\langle \cdot,\cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$.
$(x, y) \longmapsto {}^t x A y$

$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

(1) $\langle \cdot,\cdot \rangle$ est un produit scalaire.

bil. ✓ sym ${}^t x A y \in \mathbb{R}$:

$$\langle x, y \rangle = {}^t x A y = {}^t({}^t x A y) = {}^t y \, {}^t A \, x \underset{{}^t A = A}{=} {}^t y A x = \langle y, x \rangle$$

Défini positif :

$$\langle X, X \rangle = {}^t X A X = (x_1 \ x_2 \ x_3) \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= (x_1 \ x_2 \ x_3) \begin{pmatrix} x_1 + x_2 \\ x_1 + 2x_2 \\ 3x_3 \end{pmatrix} = x_1^2 + x_1 x_2 + x_2 x_1 + 2x_2^2 + 3x_3^2$$

$$= x_1^2 + 2x_1 x_2 + x_2^2 + x_2^2 + 3x_3^2$$

$$= (x_1 + x_2)^2 + x_2^2 + 3x_3^2 \geq 0$$

$\langle x, x \rangle = 0 \iff x_1 + x_2 = 0 , x_2 = 0 , x_3 = 0 \implies x_1 = x_2 = x_3 = 0.$ ✓

$F = \mathrm{Vect}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right)$, $G = \mathrm{Vect}\begin{pmatrix}0\\1\\0\end{pmatrix}$

Donner une base de $F^\perp$ et $G^\perp$ pour $\langle x, y\rangle = {}^t x A y$.

$F^\perp$: $\begin{pmatrix}x\\y\\z\end{pmatrix} \in F^\perp \Leftrightarrow (x\ y\ z) A \begin{pmatrix}1\\0\\0\end{pmatrix} = 0$ et $(x\ y\ z) A \begin{pmatrix}1\\0\\1\end{pmatrix} = 0$

$0 = (x\ y\ z)\begin{pmatrix}1 & 1 & 0\\1 & 2 & 0\\0 & 0 & 3\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}$ | $0 = (x\ y\ z)\begin{pmatrix}1\\1\\3\end{pmatrix}$

$= (x\ y\ z)\begin{pmatrix}1\\1\\0\end{pmatrix} = x+y$ | $0 = x+y+3z$

$F^\perp$

$\boxed{x+y=0,\ x+y+3z=0}$

$F^\perp = \mathrm{Vect}\begin{pmatrix}1\\-1\\0\end{pmatrix}$

$G^\perp$: $\begin{pmatrix}x\\y\\z\end{pmatrix} \in G^\perp \Leftrightarrow (x\ y\ z) A \begin{pmatrix}0\\1\\0\end{pmatrix} = 0$

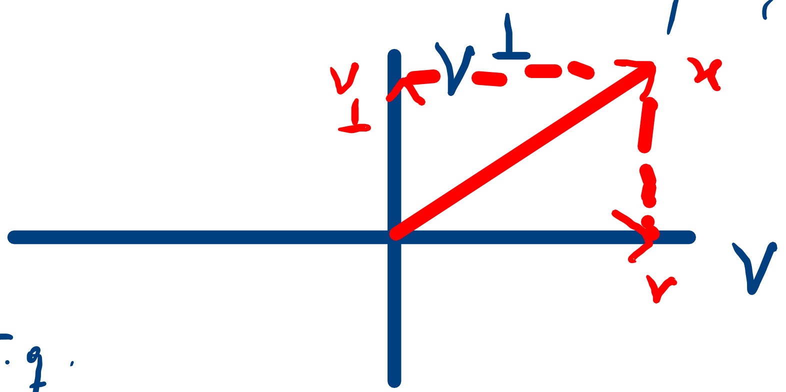
$(x\ y\ z)\begin{pmatrix}1\\2\\0\end{pmatrix} = 0$

$x + 2y = 0$

$G^\perp : \mathrm{Vect}\left(\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}-2\\1\\0\end{pmatrix}\right)$

Projections orthogonales.

$\mathbb{R}^2$, $\langle\, ,\, \rangle$ standard



$x = v + v_\perp$

$$E = V \oplus V^\perp$$

$\forall x \in E,\ \exists!\ (v, v_\perp) \in V \times V^\perp$ t.q.

$$x = v + v_\perp$$

$$p_V : \begin{array}{ccc} E & \to & E \\ x & \mapsto & v \end{array} \qquad p_{V^\perp} : \begin{array}{ccc} E & \to & E \\ x & \mapsto & v_\perp \end{array}$$

$$\begin{aligned} x &= v + v_\perp \\ &= p_V(x) + p_{V^\perp}(x) \end{aligned}$$

$$id_E = p_V + p_{V^\perp}$$

$$p_V(p_{V^\perp}(x)) = p_V(v_\perp) = p_V(0 + v_\perp) = 0$$

De même, $p_{V^\perp}(p_V(x)) = 0$

$$\begin{aligned} p_V(p_V(x)) = p_V(v) &= p_V(v+0) \\ &= v = p_V(x) \end{aligned}$$

$$p_V^2 = p_V$$

De même, $p_{V^\perp}^2 = p_{V^\perp}$

Exo 3. $u \in E\setminus\{0\}$, $V = \text{vect}(u)$

$$E = V \oplus V^\perp$$

$$x = \underbrace{\frac{\langle x,u\rangle}{\langle u,u\rangle} u}_{\in V} + \underbrace{x - \frac{\langle x,u\rangle}{\langle u,u\rangle} u}_{\in V^\perp}$$

$$p_V(x) = \frac{\langle x,u\rangle}{\langle u,u\rangle} u$$

1. Ici $E = \mathbb{R}^2$, $\langle , \rangle$ standard

$u = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $V = \text{Vect}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$p_V\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{(x_1\ x_2)\begin{pmatrix} 2 \\ -1 \end{pmatrix}}{(2\ -1)\begin{pmatrix} 2 \\ -1 \end{pmatrix}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{2x_1 - x_2}{5}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

base $\mathcal{B} = \left(e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$. $p_V(e_1) = \frac{2}{5}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $p_V(e_2) = -\frac{1}{5}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

$$P_1 = [p_V]_{\mathcal{B}} = \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$$ (la matrice de p_V dans $\mathcal{B}$)

$u' = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ est $\perp$ à u : $\langle u', u\rangle = (1\ 2)\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 0$

$V^\perp = \text{Vect}(u')$, $$p_{V^\perp}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{(x_1\ x_2)\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{(1\ 2)\begin{pmatrix} 1 \\ 2 \end{pmatrix}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{x_1 + 2x_2}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$p_{V^\perp}(e_1) = \frac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $p_{V^\perp}(e_2) = \frac{2}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$P_2 = [p_{V^\perp}]_{\mathcal{B}} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}$$

2. On a bien

$$P_1 + P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P_1 P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

matrice $n1 \times 1n = nn$

Exo 4. $\mathbb{R}^n$

$$P = \frac{a\,{}^t a}{{}^t a\, a} = \frac{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} [a_1 \cdots a_n]}{[a_1 \cdots a_n]\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}$$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$$

matrice $1n \times n1 = 11$

$$P^2 = PP = \frac{a\,{}^t a}{{}^t a\, a}\,\frac{a\,{}^t a}{{}^t a\, a} = \frac{a\,(\cancel{{}^t a a})\,{}^t a}{{}^t a a\;\cancel{{}^t a a}} = \frac{a\,{}^t a}{{}^t a\, a} = P$$

$\in \mathbb{R}$

Rappel : $M = (m_{ij}) \in M_n(\mathbb{R})$

$$\operatorname{tr} M = \sum_{i=1}^{n} m_{ii} \quad (\text{la trace})$$

$$P = \frac{\begin{bmatrix} a_1^2 & a_1 a_2 & \cdots & a_{1n} \\ a_2 a_1 & a_2^2 & a_2 a_3 \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_n a_1 & \cdots & & a_n^2 \end{bmatrix}}{a_1^2 + a_2^2 + \cdots + a_n^2}$$

$$\operatorname{tr} P = \frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i^2} = 1$$

Qui est P ?

Soit
$$p : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \longmapsto p(b) = P \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

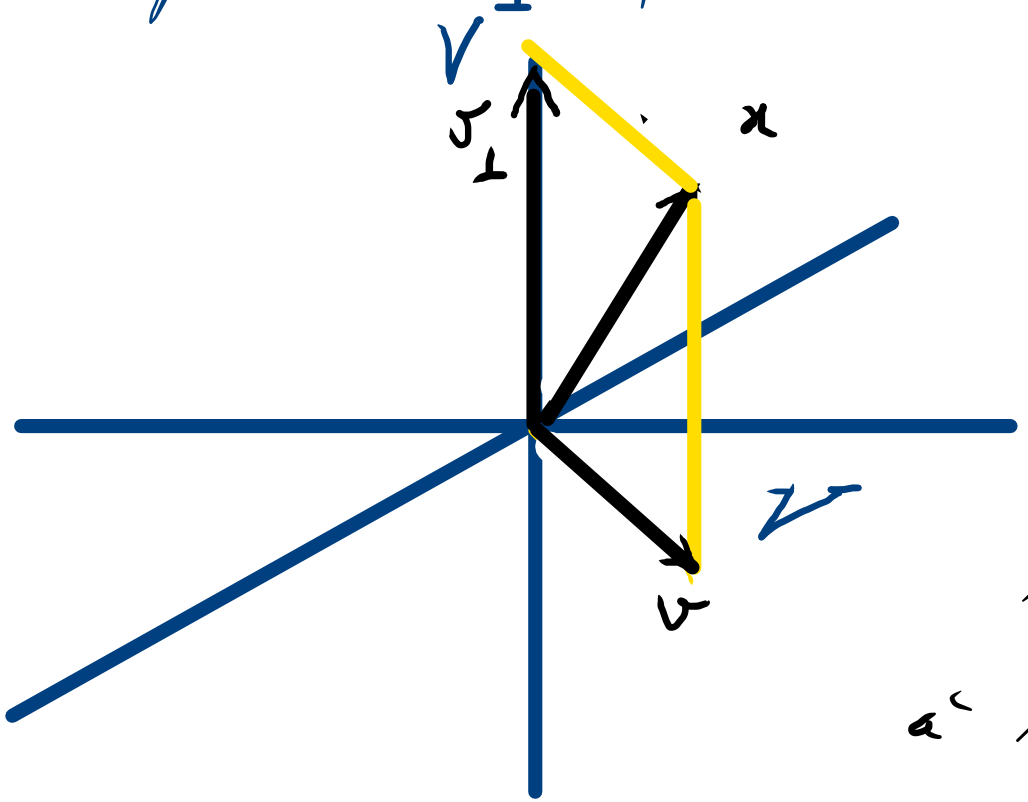
l'endomorphisme dont la matrice dans la base canonique est P

$$p(b) = \frac{a\,{}^t a}{{}^t a a}\, b = a \frac{({}^t a b)}{{}^t a a}$$

$$= a \frac{[a_1 \cdots a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}{[a_1 \cdots a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}} = \frac{\langle a, b \rangle}{\langle a, a \rangle}\, a$$

p est la projection orthogonale sur a pour le produit scalaire standard.

Remarque sur les projections orthogonales :



$$x = v + v_\perp$$

$$= p_V(x) + p_{V^\perp}(x)$$

Dans ce cours, ne pas hésiter à faire une figure.

$$V = \mathrm{Vect}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right)$$